# A predicative variant of the effective topos

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# From the Minimalist Foundation to a predicative tripos

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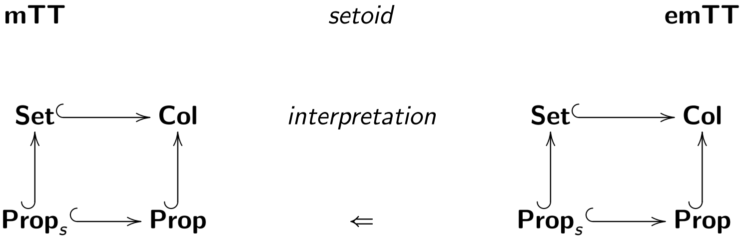

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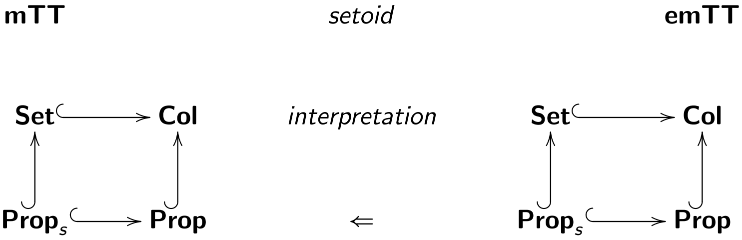


**idea**: mimic this structure to define a predicative version of the effective topos from a predicative effective tripos.

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$\mathcal{C}_r$ is a **finitely complete weakly locally cartesian closed** category with parameterized list objects.

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Define over $\mathcal{C}_r$ an indexed category representing dependent collections

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**different** from the tripos defining the effective topos, where $x$ is **not** a natural number in general

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and small propositions $\overline{\mathbf{Prop}_s}^r$ as a doctrine over the base category.

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$\alpha \in \overline{\mathbf{p}}(A)$ iff $\alpha \in \mathbf{p}(|A|)$ and $\alpha(x) \wedge x \sim_A y \vdash_{\mathbf{p}} \alpha(y)$
order is inherited from $\mathbf{p}(|A|)$.

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# Relation with the Effective Topos

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**Assemblies** (**Asm**):

objects: $(A, P)$, $A$ set, $P : A \to \mathcal{P}(\mathbb{N})$, $P(a) \neq \varnothing$ for every $a \in A$.

arrows $f : (A, P) \to (B, Q)$, $f : A \to B$ and there is $r \in \mathbb{N}$ such that $\{r\}(n) \in Q(f(a))$ for every $a \in A$ and $n \in P(a)$.

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**Rec** isomorphic to the category of subsets of natural numbers and (restrictions of) recursive functions between them.

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$\mathcal{C}_r$ is a rendering of **Rec** in $\widehat{ID_1}$ and $\mathbf{Prop}^r$ is equivalent to **wSub**:
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The effective topos $\mathcal{E}ff$ can be introduced in three different ways:

1. via tripos-to-topos construction (Hyland, Johnstone, Pitts) from $\mathbf{p}_{Eff} : \mathbf{Set}^{op} \rightarrow \mathbf{preHeyt}$; with **Set** from ZFC;
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# References


1. M. Hyland, P. Johnstone, A. Pitts. Tripos theory, 1980.
2. M. Hyland. The Effective Topos, 1981.
3. M. E. Maietti, G. Sambin. Towards a minimalist foundation for constructive mathematics, 2005.
4. M. E. Maietti. A minimalist two-level foundation for constructive mathematics, 2009.
5. M. E. Maietti, G. Rosolini. Quotient completion for the foundation of constructive mathematics, 2013.
6. H.Ishihara, M.E.Maietti, S. Maschio, T.Streicher. Consistency of the intensional level of the Minimalist Foundation with Church's Thesis and Axiom of Choice, to appear in Archive for Mathematical Logic.
7. M. E. Maietti, S. Maschio. A strictly predicative variant of Hyland's Effective Topos, submitted.