Subminimal Logics and Relativistic Negation

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Outline

1. Background
    - Minimal Logic
    - Subminimal Logics
2. Logics with Relativistic Negation
    - Axiom An^- and $\mathbf{An^-PC}$
    - Semantics of $\mathbf{An^-PC}$
    - Axiom LP and **LPPC**
    - Some More Logics with Relativistic Negation
3. Outlook

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Languages

Definition ($\mathcal{L}^+, \mathcal{L}_\bot, \mathcal{L}_\neg$)

We shall use the following propositional languages:

$$\mathcal{L}^+ ::= p|A \wedge B|A \vee B|A \to B|$$
$$\mathcal{L}_\bot ::= p|A \wedge B|A \vee B|A \to B|\bot$$
$$\mathcal{L}_\neg ::= p|A \wedge B|A \vee B|A \to B|\neg A$$

- In $\mathcal{L}_\bot$, we take $\neg A$ to be the abbreviation for $A \to \bot$.

Minimal/Intuitionistic Logic

Definition ($\mathbf{MPC}_\bot$, $\mathbf{IPC}_\bot$)

$\mathbf{MPC}_\bot$ is the smallest set of formulas of $\mathcal{L}_\bot$ containing the axioms below. Plus: If $A, A \to B \in \mathbf{MPC}_\bot$ then $B \in \mathbf{MPC}_\bot$ (MP).

Axioms

$A \to (B \to A)$; $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$;
$A \to (A \vee B)$; $B \to (A \vee B)$;
$(A \to C) \to ((B \to C) \to (A \vee B \to C))$;
$A \wedge B \to A$; $A \wedge B \to B$; $A \to (B \to (A \wedge B))$.

$\mathbf{IPC}_\bot$ in addition contains the axiom EFQ: $\bot \to A$.

- $\bot$ in $\mathbf{MPC}_\bot$ behaves like a propositional variable.

Negation and Contradiction

Definition ($\mathbf{MPC}_\neg$)

$\mathbf{MPC}_\neg$ is the smallest set of formulas of $\mathcal{L}_\neg$ containing the axioms below. Plus: If $A, A \to B \in \mathbf{MPC}_\neg$ then $B \in \mathbf{MPC}_\neg$.

Axioms

$A \to (B \to A)$; $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$;
$A \to (A \vee B)$; $B \to (A \vee B)$;
$(A \to C) \to ((B \to C) \to (A \vee B \to C))$;
$A \wedge B \to A$; $A \wedge B \to B$; $A \to (B \to (A \wedge B))$;

M: $[(A \to B) \wedge (A \to \neg B)] \to \neg A$

Call the negation-less($\mathcal{L}^+$) fragment of $\mathbf{MPC}_\neg$ as **PPC**.

Counter-intuitive Inferences Involving Negation

Definition (EFQ, NeF)

EFQ: $(A \wedge \neg A) \rightarrow B$ [for $\mathrm{MPC}_{\neg}$]
NeF: $(A \wedge \neg A) \rightarrow \neg B$

- EFQ: holds in intuitionistic logic.
- NeF: holds in minimal and intuitionistic logic.

They are seen as unsatisfactory from the criteria of:

- (Relevance) *Premises and the conclusions are related.*
- (Paraconsistency) *Contradictions do not trivialise the logic.*

Paths to Subminimality

- This motivates the study of logics with a weaker negation.
- We can weaken $\mathbf{MPC}_{\bot}$ or $\mathbf{MPC}_{\neg}$.
- $\mathbf{MPC}_{\bot}$: no axiom for $\bot \Rightarrow$ *difficult to weaken*
  $\mathbf{MPC}_{\neg}$: has the axiom M $\Rightarrow$ *amendable with weaker negation axioms*
- Such axioms are called subminimal axioms, and the logics with them (defined over **PPC**) subminimal logics.

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Known Subminimal Axioms

Definition (Co, An, NeF, N)

Colacito, de Jongh and Vargas (2017) studied the following subminimal axioms.

Co: $(A \to B) \to (\neg B \to \neg A)$; An: $(A \to \neg A) \to \neg A$;
NeF: $(A \wedge \neg A) \to \neg B$; N: $(A \leftrightarrow B) \to (\neg A \leftrightarrow \neg B)$;

Proposition (Colacito (2016), Colacito et al.(2017))

(i) Co $\Rightarrow$ NeF, Co $\Rightarrow$ N
(ii) An+N $\Leftrightarrow$ M
(iii) Co $\Rightarrow \neg\neg\neg A \to \neg A$

- Call **PPC**+N (Co) as **NPC** (**CoPC**); **NPC**+NeF as **NeFPC**.
- **NPC** is taken as the basic subminimal logic.

Graphical Representation

Logic	*Negation Axiom(s)*
$\mathbf{MPC}_\neg$	N + An: $(A \to \neg A) \to \neg A$
CoPC	Co: $(A \to B) \to (\neg B \to \neg A)$
NeFPC	N + NeF: $(A \wedge \neg A) \to \neg B$
NPC	N: $(A \leftrightarrow B) \to (\neg A \leftrightarrow \neg B)$

Question

Is there a logic between $\mathbf{MPC}_\neg$ and **CoPC**?

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An$^-$: A Weaker Version of An

Definition (An$^-$)

An$^-$: $(A \to \neg A) \to (\neg B \to \neg A)$

We define **An$^-$PC** as **NPC** $+$ An$^-$.

Proposition (separating **An$^-$PC** from **CoPC** [N.])

(i) **An$^-$PC** $\vdash$ Co; **CoPC** $\nvdash$ An$^-$.
(ii) **An$^-$PC** $\supsetneq$ **CoPC**.

Hence **CoPC** is not maximal.

Proposition (some properties of **An$^-$PC** [N.])

An$^-$PC $\nvdash A \to \neg\neg A$; **An$^-$PC** $\vdash \neg A \to \neg\neg\neg A$.

Sequent Calculus for **An$^-$PC**

Definition (Sequent Calculus **GAn$^-$** for **An$^-$PC**)

Axioms: Ax: $p \Rightarrow p$ (R$\top$: $\Gamma \Rightarrow \top$)

Rules for positive connectives:

$$\text{L}\wedge: \frac{\Gamma, A, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C} \qquad \text{R}\wedge: \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\text{L}\vee: \frac{\Gamma, A \Rightarrow C \qquad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C} \qquad \text{R}\vee: \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2} \ (i \in \{1, 2\})$$

$$\text{L}\to: \frac{\Gamma, A \to B \Rightarrow A \qquad \Gamma, B \Rightarrow C}{\Gamma, A \to B \Rightarrow C} \qquad \text{R}\to: \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}$$

Rules for negation:

$$\text{N}: \frac{\Gamma, \neg A, A \Rightarrow B \qquad \Gamma, \neg A, B \Rightarrow A}{\Gamma, \neg A \Rightarrow \neg B} \qquad \text{An}^- : \frac{\Gamma, \neg B, A \Rightarrow \neg A}{\Gamma, \neg B \Rightarrow \neg A}$$

Cut and Equivalence with Hilbert-system

We will in addition consider the following rule.

Definition (Cut)

$$\text{Cut: } \frac{\Gamma \Rightarrow A \qquad \Gamma', A \Rightarrow B}{\Gamma, \Gamma' \Rightarrow B}$$

It is straightforward to establish the following equivalence:

Proposition (equivalence with $\mathbf{An^-PC}$ [N.])

$\Gamma \vdash_{\mathbf{An}^-} A$ *if and only if* $\vdash_{\mathbf{GAn}^- + \mathrm{Cut}} \Gamma \Rightarrow A$

A Characterisation of $\mathbf{An^-PC}$

Definition (classes F^+/F^-)

$$F^+ ::= p|P_1 \wedge P_2|P \vee A|A \vee P|A \to P|N \to A$$
$$F^- ::= \neg A|N \wedge A|A \wedge N|N_1 \vee N_2|P \to N$$
$$(P \in F^+, N \in F^-, A \in F^+ \cup F^-)$$

Proposition (separating $\mathbf{An^-PC}$ from $\mathbf{MPC}_\neg$ [N.])

(i) If $\vdash_{\mathbf{GAn^-}+\mathrm{Cut}} \Gamma \Rightarrow A$ and $A \in F^-$, then Γ has a formula in F^-.
(ii) $\nvdash_{\mathbf{GAn^-}+\mathrm{Cut}} \Rightarrow \neg A$ for any A; hence $\mathbf{MPC}_\neg \supsetneq \mathbf{An^-PC}$.

- To see the last part, recall e.g. $\vdash_{\mathbf{M}} \Rightarrow \neg\neg(p \to p)$.
- Negation in $\mathbf{An^-PC}$ is *relativistic*, in the sense of (ii).

Graphical Representation

Logic	*Negation Axiom(s)*
$\mathbf{MPC}_{\neg}$	N + An: $(A \to \neg A) \to \neg A$
$\mathbf{An^{-}PC}$	N + An^{-}: $(A \to \neg A) \to (\neg B \to \neg A)$
CoPC	Co: $(A \to B) \to (\neg B \to \neg A)$
NeFPC	N + NeF: $(A \wedge \neg A) \to \neg B$
NPC	N: $(A \leftrightarrow B) \to (\neg A \leftrightarrow \neg B)$

-All subminimal extensions of $\mathbf{An^{-}PC}$ have relativistic negation.

Further Proof-theoretic Properties of $\mathbf{An}^-\mathbf{PC}$

Cut turns out to be admissible in $\mathbf{GAn}^-$:

Proposition (N.)

(i) If $\vdash_{\mathbf{GAn}^- + \mathrm{Cut}} \Gamma \Rightarrow A$ then $\vdash_{\mathbf{GAn}^-} \Gamma \Rightarrow A$
(ii) $\mathbf{An}^-\mathbf{PC}$ is decidable.

As further consequences of cut-admissibility,

- We can show the disjunction property of $\mathbf{An}^-\mathbf{PC}$;
- The interpolation theorem holds for $\mathbf{An}^-\mathbf{PC}$, extending the result of Colacito (2016) on **NPC**.

Outline

$\mathbf{MPC}_\neg$ and Kripke Semantics

Definition (Kripke semantics for $\mathbf{MPC}_\neg$)

A minimal frame is a triple $(W, \leq, F)$.

- $(W, \leq)$ is a poset.
- $F \subseteq W$ is an upward closed set;
  i.e. $w \in F$ and $w' \geq w$ implies $w' \in F$.

We have the following valuation of negation.
- $\mathcal{M}, w \Vdash \neg A \Leftrightarrow \forall w' \geq w[\mathcal{M}, w' \Vdash A \Rightarrow w' \in F]$

- F denotes the set of worlds where *all* negations hold.

An$^-$PC and Kripke Semantics

Definition (Kripke semantics for An$^-$PC)

An An$^-$-frame is a quadruple $(W, \leq, F, G)$.

- $(W, \leq)$ is a poset.
- $F, G \subseteq W$ are upward closed subsets s.t. $F \subseteq G$;

We have the following valuation of negation.

- $\mathcal{M}, w \Vdash \neg A \Leftrightarrow \forall w' \geq w[\mathcal{M}, w' \Vdash A \Rightarrow w' \in F] \wedge w \in G$

- G denotes the set of worlds where *some* negations hold.
- Thus G is a natural counterpart of F.
- $G \backslash F$ is the area where negations hold untrivially.

Completeness of **An$^-$PC**

The following properties hold with repect to the semantics.

Proposition (completeness of An$^-$PC [N.])

$\Gamma \vdash_{\mathbf{An}^-} A \Leftrightarrow \Gamma \vDash_{\mathbf{An}^-} A$

Proposition (finite model property for An$^-$PC [N.])

An$^-$PC is weakly complete with respect to the class of finite An$^-$-frames.

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Axiom LP

Q. Is there a logic between $\mathbf{MPC}_\neg$ and $\mathbf{An^-PC}$?

Definition (LP)

LP: $(A \leftrightarrow \neg A) \rightarrow \neg A$

- LP can be seen as expressing the liar's paradox.

Proposition (class of LP-frames [N.])

Let $\mathcal{F}$ be an An^--frame. Then:
$\mathcal{F} \vDash \text{LP} \Leftrightarrow \forall w \in W[w \in G \vee \exists w' \geq w(w' \in G \backslash F)]$

- The frame property says you will eventually arrive in G.
- If we take $G = \emptyset$, then LP is not valid in the frame.
- Thus by soundness, LP is not a theorem of $\mathbf{An^-PC}$.

LPPC

Definition (LPPC)

We define $\mathbf{LPPC} := \mathbf{An^-PC} + \mathrm{LP}$

- By the previous proposition, $\mathbf{LPPC} \supsetneq \mathbf{An^-PC}$.

Proposition (N.)

LPPC is sound and complete with the class of LP-frames.

- Any LP-frame with $G \subsetneq W$ can refute An; so $\mathbf{MPC}_\neg \supsetneq \mathbf{LPPC}$.
- **LPPC** satisfies the disjunction property and the finite model property.

Graphical Representation

Logic	*Negation Axiom(s)*
$\mathbf{MPC}_{\neg}$	N + An: $(A \to \neg A) \to \neg A$
LPPC	N + An^{-} + LP: $(A \leftrightarrow \neg A) \to \neg A$
$\mathbf{An^{-}PC}$	N + An^{-}: $(A \to \neg A) \to (\neg B \to \neg A)$
CoPC	Co: $(A \to B) \to (\neg B \to \neg A)$
NeFPC	N + NeF: $(A \wedge \neg A) \to \neg B$
NPC	N: $(A \leftrightarrow B) \to (\neg A \leftrightarrow \neg B)$

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Countably Many Logics with Relativistic Negation

Formulas below set the maximal length of intuitionistic frames.

Proposition (a result from intermediate logics)

Let $\mathbf{bd_1} := p_1 \vee \neg p_1$, $\mathbf{bd}_{n+1} := p_{n+1} \vee (p_{n+1} \to \mathbf{bd}_n)$.
Then $\mathcal{F} \vDash_{\mathbf{I}} \mathbf{bd}_i \Leftrightarrow W$ does not have chains of $> i$ worlds.

We can apply this to the length of chains in $W \backslash G$ in LP-frame.

Proposition (N.)

Let $\mathbf{Gd}_1 := (p_1 \to \neg p_1) \to \neg p_1$, $\mathbf{Gd}_{n+1} := p_{n+1} \vee (p_{n+1} \to \mathbf{Gd}_n)$.
Then $\mathcal{F} \vDash_{\mathbf{LP}} \mathbf{Gd}_i \Leftrightarrow W \backslash G$ does not have chains of $\geq i$ worlds.

With this it is easy to verify (via soundness),
$\mathbf{MPC}_\neg = \mathbf{LPPC} + \mathbf{Gd}_1 \supsetneq \mathbf{LPPC} + \mathbf{Gd}_2 \supsetneq \ldots \supsetneq \mathbf{LPPC}$.

Future Directions

- Is there a maximal subminimal logic with relativistic negation?
- How many logics are there between $\mathbf{MPC}_\neg$ and $\mathbf{An}^-\mathbf{PC}$? (Bezhanishvili, Colacito and de Jongh (2017) showed uncountably many exist between $\mathbf{MPC}_\neg$ and **NPC**.)
- How does our semantics correspond with the Kripke semantics of Colacito et al.(2017)?

Reference

- N. Bezhanishvili, A. Colacito and D. de Jongh (2017). *A lattice of subminimal logics of negation*. TbiLLC 2017: Twelfth International Tbilisi Symposium on Language, Logic and Computation.
- A. Colacito (2016). *Minimal and Subminimal Logic of Negation*. Master's Thesis. University of Amsterdam.
- A. Colacito, D. de Jongh and A.L. Vargas (2017). *Subminimal negation*. Soft Computing 21: 165-174.

Reference

- J. van Benthem (1984). *Correspondence Theory*. in D. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic, Volume II: Extensions of Classical Logic*. Kluwer Academic Publishers.
- A. Chagrov and M. Zakharyaschev (1997). *Modal Logic*. Oxford University Press.
- D. van Dalen (2013). *Logic and Structure*. Fifth edition. Springer.

Reference

- K. Sano, T. Kurahashi, T. Usuba, H. Kurokawa and M. Kikuchi (2016). *Proof and Truth in Mathematics: Modal Logic and the Foundation of Mathematics* (in Japanese). Kyoritsu Shuppan.
- A.S. Troelstra and H. Schwichtenberg (2000). *Basic Proof Theory*. Second edition. Cambridge University Press.
- A.S. Troelstra and D. van Dalen (1988). *Constructivism in Mathematics: An Introduction*. Volume I, Elsevier.