The Eigen-Distribution of Weighted AND-OR Trees

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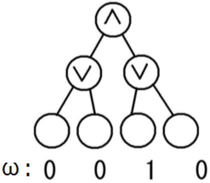
Outline

In 2007, Liu–Tanaka characterized the distribution which achieves the distributional complexity for binary AND-OR tree, and showed the uniqueness of such distribution.

The characterization is extended to multi-branching trees, but the uniqueness was not proved.

We introduce the *weighted tree*, and give a proof of the uniqueness.

An AND-OR tree is a tree whose root is labeled by AND ($\wedge$) and the internal nodes are level-by-level labeled by OR ($\vee$) or AND alternatively except for leaves.
Such a tree is also called *Game tree*.
For AND-OR tree $\mathcal{T}$, function ω from the set of all leaves of $\mathcal{T}$ to $\{0, 1\}$ is called an assignment.



An assignment is denoted by a 0-1 sequence.
(e.g $\omega = 0010$)

Algorithms

An algorithm A tells how to proceed to evaluate a tree.
Algorithms have the following properties:

- deterministic :
  The choice of leaves in each step is unique.
- depth-first :
  If an algorithm evaluates the value of some subtree, it never evaluate another subtree until it finishes to evaluates the current one.

An algorithm is *directional* if there is some linear ordering on the leaves such that the computation follows this ordering.

Example of algorithm

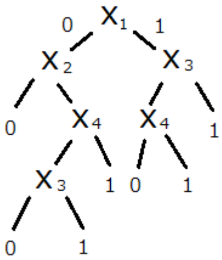


Figure: algorithm

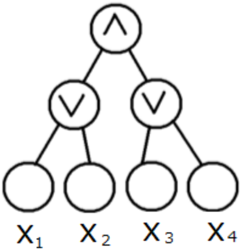
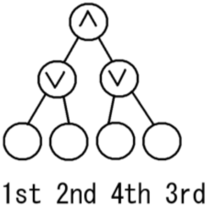


Figure: AND-OR tree

Notation

Let A be a directional algorithm whose priority of searching leaves is as follows:

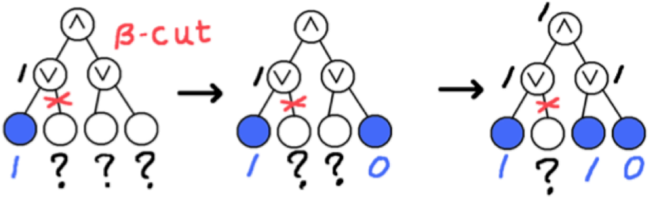


We say A =**1243**.

Given an assignment ω and an algorithm A, $C(A,\omega)$ denotes the number of leaves checked by A under ω.

How to evaluate the cost

For the directional algorithm A $=$**1243** and $\omega = 1010$



In this case, $C(A, \omega) = 3$

Let d be a (probability) distribution on the set of assignments, the expected cost of A under the distribution d is defined by

$$C(A, d) := \sum_{\omega} d(\omega) C(A, \omega)$$

Given a class of algorithms $\mathcal{A}$,
distributional complexity w.r.t $\mathcal{A}$ is defined by:

$$\max_{d} \min_{A \in \mathcal{A}} C(A, d)$$

A distribution *d* satisfying

$$\min_{A \in \mathcal{A}} C(A, d) = \max_{d} \min_{A \in \mathcal{A}} C(A, d)$$

is called *eigen-distribution* (w.r.t $\mathcal{A}$).

Liu–Tanaka gave a characterization of the eigen-distribution for uniform binary tree.

The "i-set" is the set of assignments which are difficult to evaluate.

Definition (i-set for n-branching trees)

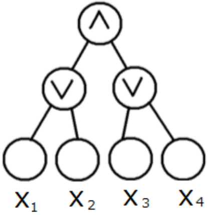
Given an n-branching tree $\mathcal{T}$, $i \in \{0, 1\}$, the i-set consists of assignments such that

• the root has value i,

• if an AND-node has value 0 (or OR-node has value 1), just one of its children has value 0 (1), and other n-1 children have 1 (0).

Example of i-set

For AND-OR tree $\mathcal{T}_2^2$,
1-set$=$
$\{1010, 1001, 0110, 0101\}$
0-set$=$
$\{1000, 0100, 0010, 0001\}$



For example, assignment 1110 $\notin$1-set.

Definition

E^1-distribution (w.r.t $\mathcal{A}$) is a distribution on 1-set whose expected cost is independent of the choice of algorithms in $\mathcal{A}$.

Theorem (Liu–Tanaka (2007))

Let $\mathcal{T}$ be a uniform binary AND-OR tree.
Then, the E^1-distribution is the unique eigen-distribution, espacially, which is the uniform distribution on 1-set.

Theorem (Suzuki–Nakamura (2012))

$\mathcal{T}$: uniform binary AND-OR tree,
$\mathcal{A}$: (closed) set of algorithms
Then, eigen-distribution is equivalent to E^1-distribution w.r.t $\mathcal{A}$.
Furthermore, if $\mathcal{A}$ is the set of all directional algorithms, the uniqueness fails.

Theorem (Peng et al. (2016))

$\mathcal{T}$: n-branching AND-OR tree
$\mathcal{A}$: (closed) set of algorithms
Then, eigen-distribution is equivalent to E^1-distribution w.r.t $\mathcal{A}$.

Theorem (Peng et al. (2016))

Let $\mathcal{T}$ be a n-branching AND-OR tree of height 2. Then E^1-distribution is the uniform distribution on 1-set.

It remains that

Theorem

For any n-branching AND-OR tree,
E^1-distribution is the uniform distribution on 1-set.

To show this theorem,
we generalize the definition of “cost”.

For any algorithm A, and assignment ω, we define
$\#_0(A,\omega) :=$ the number of leaves checked by A and assigned 0 under ω.

$\#_1(A,\omega) :=$ the number of leaves checked by A and assigned 1 under ω.

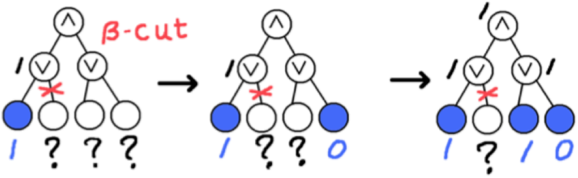
Definition

Let $a, b \in \mathbb{R}_{>0}$. The generalized cost $C(A,\omega;a,b)$ of A under ω is defined as follows:

$$C(A,\omega;a,b) := a \cdot \#_0(A,\omega) + b \cdot \#_1(A,\omega)$$

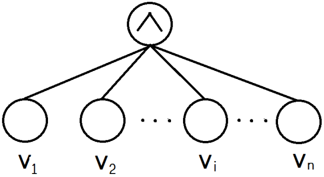
$C(A,d;a,b)$, $E^1(a,b)$-distribution,... are defined by the same way.

For the directional algorithm A =**1243** and $\omega = 1010$



In this case, $C(A, \omega; a, b) = a + 2b$

Example (AND-OR tree of height 1)



1-set $=\{\omega_0\}$, 0-set $=\{\ \omega_i \mid i = 1, 2, \cdots, n\}$

where $\omega_0(v_j) = 1, \ \ \omega_i(v_j) = \begin{cases} 1\ (i \neq j) \\ 0\ (i = j) \end{cases} \qquad (j = 1, 2, \cdots, n)$

Then,

$C(A, d_{uni}(\text{1-set}); a, b) = nb,$

$C(A, d_{uni}(\text{0-set}); a, b) = a + \frac{n-1}{2}b$

Theorem

Let $a, b \in \mathbb{R}_{>0}$. For any n-branching AND-OR tree, $E^1(a, b)$-distribution is the uniform distribution on 1-set.

The same statement hold for $E^0(a, b)$-distribution.

We prove this by induction on the height h.

Induction step

We consider the $E^1(a, b)$-distribution d and assume h is even. The proof consists of two parts.

Lemma (1)

The probability of an assignment depends only on the value of nodes in height h.

The proof is essentially the same as the case height 1.
We use the condition "*nondirectional*" here.

We define the distribution d' for AND-OR tree of height h.

$$d'(\omega') := \sum_{\omega \in \Omega_{\omega'}} d(\omega)$$

where $\Omega_{\omega'} := \{\omega \mid \omega \text{ assigns } \omega' \text{ to the nodes of height } h.\}$

By the previous lemma, d' can be represented by

$$d'(\omega') = C \cdot d(\omega)$$

We should note that the cardinality of $\Omega_{\omega'}$ is independent of ω',

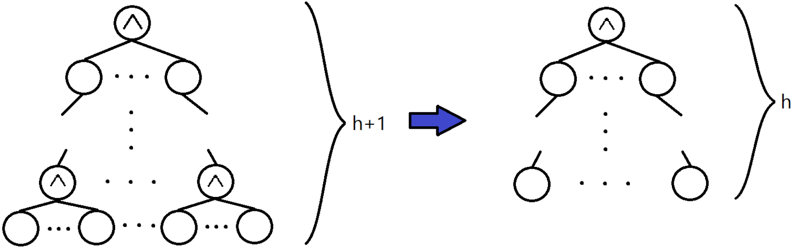
Lemma (2)

d' is an $E^1(a + \frac{n-1}{2}b, nb)$-distribution for AND-OR tree of height h.

(sketch)
Given any algorithm A for height h, we can define an A' for height $h+1$ satisfying

$$C(A', d; a, b) = C(A, d'; a + \frac{n-1}{2}b, nb)$$

Since d is $E^1(a, b)$-distribution, the claim holds.



cost : 0 ⇒ a, 1 ⇒ b   cost: 0 ⇒ a+$\frac{n-1}{2}$b, 1 ⇒ nb

$d : E^1(a, b)$-distribution $d' : E^1(a + \frac{n-1}{2}, nb)$-distribution

Recall (height 1)

$C(A, d_{uni}(0\text{-set}); a, b) = a + \frac{n-1}{2}b,$
$C(A, d_{uni}(1\text{-set}); a, b) = nb$

Theorem

Let $a, b \in \mathbb{R}_{>0}$. For any n-branching AND-OR tree, $E^1(a, b)$-distribution is the uniform distribution on 1-set.

(proof)
By induction hypothesis, d' is the uniform distribution on 1-set for height h.
Since $d = \frac{1}{c}d'$, so d is also the uniform distribution on 1-set for height $h + 1$.

□

Since eigen-distribution is equivalent to E^1-distribution, we get the uniqueness of the eigen-distribution.

Corollary

Let $\mathcal{T}$ be an n-branching AND-OR tree.
Then E^1-distribution is the uniform distribution on 1-set.

Generally, the following does NOT hold for AND-OR tree:

$$E^1(a,b)\text{-distribution} \Leftrightarrow \text{eigen-distribution}$$

For example, if height is 1

$$C(A, d_{uni}(\text{1-set}); a, b) \leq C(A, d_{uni}(\text{0-set}); a, b)$$

$$\Leftrightarrow nb \leq a + \frac{n-1}{2}b \quad \Leftrightarrow \quad \frac{n+1}{2}b \leq a$$

Moreover, if the equality holds, then there are uncountably many eigen-distributions.
So, the uniqueness of the eigen-distribution for weighted tree fails.

References

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