

Constructive strict implication

Tadeusz Litak (FAU Erlangen-Nuremberg)
and Albert Visser (Utrecht)

March 7, 2018

This talk

- Basically an advertisement for Tadeusz Litak and Albert Visser, *Lewis meets Brouwer: constructive strict implication*, *Indagationes Mathematicae*, A special issue “L.E.J. Brouwer, fifty years later”, vol. 29 (2018), no. 1, pp. 36–90, DOI: 10.1016/j.indag.2017.10.003, URL: <https://arxiv.org/abs/1708.02143>
- (same issue as Wim’s talk yesterday)
- and some of our ongoing work

- As we all know (or do we?) the following is the original syntax of modern modal logic :

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- Truth of strict implication at w = truth of material implication in all possible worlds seen from w

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- ... which did not seem to make him happy
- He didn't even like the name "modal logic" ...

There *is* a logic restricted to indicatives; the truth-value logic most impressively developed in “*Principia Mathematica*”. But those who adhere to it usually have thought of it—so far as they understood what they were doing—as being the universal logic of propositions which is independent of mode. And when that universal logic was first formulated in exact terms, they failed to recognize it as the only logic which is *independent* of the mode in which propositions are entertained and dubbed it “modal logic”.

- Curiously, Lewis was opened towards non-classical systems (mostly MV of Łukasiewicz)
 - A detailed discussion in *Symbolic Logic*, 1932
 - A paper on “Alternative Systems of Logic”, *The Monist*, same year
 - Both references analyze possible definitions of “truth-implications” / “implication-relations” available in finite, but not necessarily binary matrices.
- I found just one reference where he mentions (rather favourably) Brouwer and intuitionism . . .

[T]he mathematical logician Brouwer has maintained that the law of the Excluded Middle is not a valid principle at all. The issues of so difficult a question could not be discussed here; but *let us suggest a point of view at least something like his. . . . The law of the Excluded Middle is not writ in the heavens: it but reflects our rather stubborn adherence to the simplest of all possible modes of division, and our predominant interest in concrete objects as opposed to abstract concepts.* The reasons for the choice of our logical categories are not themselves reasons of logic any more than the reasons for choosing Cartesian, as against polar or Gaussian coördinates, are themselves principles of mathematics, or the reason for the radix 10 is of the essence of number.

- No indication he was aware of
- As we will see, maybe he should've followed up on that ...
- ...especially that there were more analogies between him and Brouwer
 - almost perfectly parallel life dates
 - wrote his 1910 PhD on *The Place of Intuition in Knowledge*
 - a solid background in/influence of idealism and Kant ...

New incarnations of strict implication

- Metatheory of arithmetic

Σ_1^0 -preservativity for a theory T extending HA:

$$A \neg_T B \Leftrightarrow \forall \Sigma_1^0\text{-sentences } S (T \vdash S \rightarrow A \Rightarrow T \vdash S \rightarrow B)$$

Albert working on this since 1985, later more contributions made also by Iemhoff, de Jongh, Zhou ...

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- Functional programming

Distinction between **arrows** of John Hughes and **applicative functors/idioms** of McBride/Patterson

A series of papers by Lindley, Wadler, Yallop

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$$A \dashv_{\exists T} B \Leftrightarrow \forall \Sigma_1^0\text{-sentences } S (T \vdash S \rightarrow A \Rightarrow T \vdash S \rightarrow B)$$

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A series of papers by Lindley, Wadler, Yallop

- Proof theory of guarded (co)recursion

Nakano and more recently Clouston&Goré

Idioms are oblivious, arrows are meticulous, monads are promiscuous

Sam Lindley, Philip Wadler and Jeremy Yallop

*Laboratory for Foundations of Computer Science
The University of Edinburgh*

Abstract

We revisit the connection between three notions of computation: Moggi's *monads*, Hughes's *arrows* and McBride and Paterson's *idioms* (also called *applicative functors*). We show that idioms are equivalent to arrows that satisfy the type isomorphism $A \rightsquigarrow B \simeq 1 \rightsquigarrow (A \rightarrow B)$ and that monads are equivalent to arrows that satisfy the type isomorphism $A \rightsquigarrow B \simeq A \rightarrow (1 \rightsquigarrow B)$. Further, idioms embed into arrows and arrows embed into monads.

Keywords: applicative functors, idioms, arrows, monads

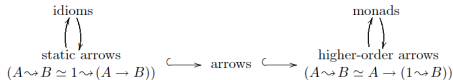


Fig. 1. Idioms, arrows and monads

\rightsquigarrow here is our \rightarrow

ENTCS 2011, proceedings of MSFP 2008

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- ...and Kripke semantics is ideal for this

Kripke semantics for intuitionistic \Box :

- Nonempty set of worlds
- Two relations:
 - Intuitionistic partial order relation \preceq , drawn as \rightarrow ;
 - Modal relation \sqsubset , drawn as \rightsquigarrow .
- Semantics for \Box : $w \Vdash \Box\phi$ if for any $v \sqsubset w$, $v \Vdash \phi$
- Semantics for \neg :

$w \Vdash \phi \neg \psi$ if for any $v \sqsubset w$, $v \Vdash \phi$ implies $v \Vdash \psi$

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upward closed?

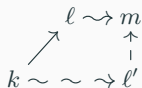
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- Is it stronger than the one ensuring persistence for $\Box A$?

Four frame conditions (known since 1980's)



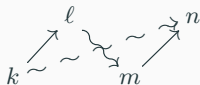
□-p

(persistence for □)



prefixing

(persistence for \neg)



mix /brilliance

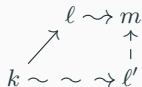


postfixing

\Leftrightarrow both are equivalent

in presence of □-p
and collapse \neg to □

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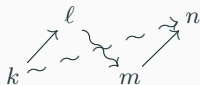
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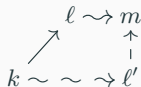
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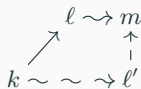
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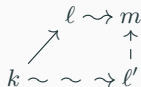
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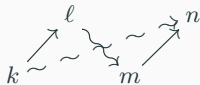
\Box -p

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- ... but $\neg\exists$ can feel it!
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- Over prefixing (or $\neg\exists$ -frames) $\Box(\phi \rightarrow \psi)$ implies $\phi \neg\exists \psi$, but not the other way around

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- ... only three consistent logics of (disjoint unions of) singleton(s) ...
- ... and yet intuitionistically you have a whole zoo: logics of (type inhabitation of) idioms, arrows, strong monads/PLL with superintuitionistic logics as a degenerate case
also recent attempts at “intuitionistic epistemic logics”, esp. Artemov and Protopopescu

Axioms and rules of iA^- :

Those of IPC plus:

$$\text{Tra} \quad \phi \multimap \psi \rightarrow \psi \multimap \chi \rightarrow \phi \multimap \chi$$

$$\text{K}_a \quad \phi \multimap \psi \rightarrow \phi \multimap \chi \rightarrow \phi \multimap (\psi \wedge \chi)$$

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Axioms and rules of the full minimal system iA :

All the axioms and rules of IPC and iA^- and

$$\text{Di} \quad \phi \multimap \chi \rightarrow \psi \multimap \chi \rightarrow (\phi \vee \psi) \multimap \chi.$$

Derivation exercises

Lots to be found in our paper, e.g., a generalization of K_a :

$$\begin{array}{ll} \phi \multimap (\psi \rightarrow \chi) \vdash (\phi \wedge \psi) \multimap (\psi \wedge (\psi \rightarrow \chi)) & \text{by } N_a \text{ and } K_a \\ \vdash (\phi \wedge \psi) \multimap \chi & \text{by monotonicity of } \multimap \end{array}$$

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We thus get

$$\psi \multimap \chi \dashv\vdash (\psi \vee \neg\psi) \multimap (\psi \rightarrow \chi)$$

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- We derived syntactically why you need IPC to get \multimap to work
- Note no other classical tautology in one variable would do:

$$p \multimap q \not\vdash (\neg\neg p \rightarrow p) \multimap (p \rightarrow q)$$

- Completeness results for many such systems published by Iemhoff et al

Her 2001 PhD, 2003 MLQ, 2005 SL with de Jongh and Zhou

Also Zhou's ILLC MSc in 2003

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- In our paper, we announce more such completeness and correspondence results

based on on a suitable extension of Gödel-McKinsey-Tarski and

Wolter-Zakharyashev for ordinary intuitionistic modal logics

Details to be published separately

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...
- ...or even more generally, that of arithmetical interpretation of a propositional logic

- Extend \mathcal{L} to $\mathcal{L}_{\odot_0, \dots, \odot_k}$ with operators \odot_0, \dots, \odot_k where \odot_i has arity n_i
- F assigns to every \odot_i an arithmetical formula $A(v_0, \dots, v_{n_i-1})$ where all free variables are among the variables shown
- We write $\odot_{i,F}(B_0, \dots, B_{n_i-1})$ for $F(\odot_i)(\ulcorner B_0 \urcorner, \dots, \ulcorner B_{n_i-1} \urcorner)$
Here $\ulcorner C \urcorner$ is the numeral of the Gödel number of C
- f maps $Vars$ to arithmetical sentences. Define $(\phi)_F^f$:
 - $(p)_F^f := f(p)$
 - $(\cdot)_F^f$ commutes with the propositional connectives
 - $(\odot_i(\phi_0, \dots, \phi_{n_i-1}))_F^f := \odot_F((\phi_0)_F^f, \dots, (\phi_{n_i-1})_F^f)$

- Let T be an arithmetical theory

An extension of i-EA, the intuitionistic version of Elementary Arithmetic, in the arithmetical language

- A modal formula in $\mathcal{L}_{\odot_0, \dots, \odot_k}$ is **T -valid** w.r.t. F iff, for all assignments f of arithmetical sentences to $Vars$, we have $T \vdash (\phi)_F^f$.
- Write $\Lambda_{T,F}$ for the set of $\mathcal{L}_{\odot_0, \dots, \odot_k}$ -formulas that are T -valid w.r.t. F .
- Of course, $\Lambda_{T,F}$ interesting only for well-chosen F

- First, consider a single unary $\odot = \Box$ and any arithmetical theory T ...
- ... which comes equipped with a $\Delta_0(\text{exp})$ -predicate α_T encoding its axiom set.
- Let provability in T be arithmetised by prov_T .
- Set $F_{0,T}(\Box) := \text{prov}_T(v_0)$. Let $\Lambda_T^* := \Lambda_{T,F_{0,T}}$.
- Intuitionistic Löb's logic i-GL is given by the following axioms over IPC.

$$\mathbf{N} \vdash \phi \Rightarrow \vdash \Box\phi$$

$$\mathbf{K} \vdash \Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$$

$$\mathbf{L} \vdash \Box(\Box\phi \rightarrow \phi) \rightarrow \Box\phi$$

The theory **GL** is obtained by extending **i-GL** with classical logic

If T is a Σ_0^1 -sound classical theory, then $\Lambda_T^* = \mathbf{GL}$ (Solovay)

In contrast, the logic **i-GL** is not complete for **HA**:

- $\vdash \Box \neg \neg \Box \phi \rightarrow \Box \Box \phi$.
- $\vdash \Box (\neg \neg \Box \phi \rightarrow \Box \phi) \rightarrow \Box \Box \phi$
- $\vdash \Box (\phi \vee \psi) \rightarrow \Box (\phi \vee \Box \psi)$.

Still unknown what the ultimate axiomatization is

- Many possible interpretations of a binary connective
not all of them producing Lewis' arrows!
 - Interpretability
 - Π_1^0 -conservativity
 - Σ_1^0 -preservativity
- classically, the last two intertranslatable, like \Box and \Diamond

- The notion of Σ_1^0 -preservativity for a theory T (Visser 1985) is defined as follows:
- $A \dashv\vdash_T B$ iff, for all Σ_1^0 -sentences S , if $T \vdash S \rightarrow A$, then $T \vdash S \rightarrow B$
- This does yields Lewis' arrow ...
- ...with interesting additional axioms

Examples of valid principles

$$4_a \vdash \phi \rightarrow \Box \phi$$

$$L_a (\Box \phi \rightarrow \phi) \rightarrow \phi$$

$$W_a (\phi \wedge \Box \psi) \rightarrow \psi \rightarrow \phi \rightarrow \psi$$

$$W'_a \phi \rightarrow \psi \rightarrow (\Box \psi \rightarrow \phi) \rightarrow \psi$$

$$M_a \phi \rightarrow \psi \rightarrow (\Box \chi \rightarrow \phi) \rightarrow (\Box \chi \rightarrow \psi)$$

$$M'_a (\phi \wedge \Box \chi) \rightarrow \psi \rightarrow \phi \rightarrow (\Box \chi \rightarrow \psi)$$

- Still no ultimate axiomatization... but perhaps better candidates and better insights than for \Box only, see our paper

Additional axioms in well-behaved/pathological theories

E.g., in presence of **The Completeness Principle** for a theory T :

$$S_a (\phi \rightarrow \psi) \rightarrow \phi \rightarrow \psi, \quad \text{i.e., } S'_a: \phi \rightarrow \psi \rightarrow \phi \rightarrow \Box \psi$$

- Our present work includes computation of fixpoints of modalized formulas
- (below Sa , is more interesting than in presence of \square only!)
- ...encoding of fixpoints of positive formulas and retraction of μ -calculus

- Happy birthday to you, Ishihara-sensei!
- A word from Albert:
A very nice program and some well-known speakers.
- He asked me to pass his greetings to numerous friends here