Constructive strict implication

Tadeusz Litak (FAU Erlangen-Nuremberg)
and Albert Visser (Utrecht)

March 7, 2018

This talk

- Basically an advertisement for
  Tadeusz Litak and Albert Visser, *Lewis meets Brouwer: constructive strict implication*, Indagationes Mathematicae, A special issue "L.E.J. Brouwer, fifty years later", vol. 29 (2018), no. 1, pp. 36–90, DOI: 10.1016/j.indag.2017.10.003, URL: `https://arxiv.org/abs/1708.02143`
- (same issue as Wim's talk yesterday)
- and some of our ongoing work

- As we all know (or do we?) the following is the original syntax of modern modal logic :

$$\mathcal{L}_{\dashv 3} \quad \phi, \psi ::= \top \mid \bot \mid p \mid \phi \to \psi \mid \phi \vee \psi \mid \phi \wedge \psi \mid \phi \dashv 3\, \psi$$

- As we all know (or do we?) the following is the original syntax of modern modal logic :

$$\mathcal{L}_{\strictif} \quad \phi, \psi ::= \top \mid \bot \mid p \mid \phi \to \psi \mid \phi \vee \psi \mid \phi \wedge \psi \mid \phi \strictif \psi$$

- $\strictif$ is the strict implication of Clarence Irving Lewis (1918,1932)

  who is not C.S. Lewis, David Lewis or Lewis Carroll

- As we all know (or do we?) the following is the original syntax of modern modal logic :

$$\mathcal{L}_{\prec} \quad \phi, \psi ::= \top \mid \bot \mid p \mid \phi \to \psi \mid \phi \vee \psi \mid \phi \wedge \psi \mid \phi \prec \psi$$

- $\prec$ is the strict implication of Clarence Irving Lewis (1918,1932)

  who is not C.S. Lewis, David Lewis or Lewis Carroll

- $\Box\phi$ is then definable ...

- As we all know (or do we?) the following is the original syntax of modern modal logic :

$$\mathcal{L}_{\multimap} \quad \phi, \psi ::= \top \mid \bot \mid p \mid \phi \to \psi \mid \phi \vee \psi \mid \phi \wedge \psi \mid \phi \multimap \psi$$

- $\multimap$ is the strict implication of Clarence Irving Lewis (1918,1932)

  who is not C.S. Lewis, David Lewis or Lewis Carroll

- $\Box\phi$ is then definable ...
- ... as $\top \multimap \phi$. Over the classical propositional calculus, the converse holds too ...

- As we all know (or do we?) the following is the original syntax of modern modal logic :

$$\mathcal{L}_{\prec\!\!3} \quad \phi, \psi ::= \top \mid \bot \mid p \mid \phi \to \psi \mid \phi \vee \psi \mid \phi \wedge \psi \mid \phi \prec\!\!3\, \psi$$

- $\prec\!\!3$ is the strict implication of Clarence Irving Lewis (1918,1932)

  who is not C.S. Lewis, David Lewis or Lewis Carroll

- $\Box\phi$ is then definable ...
- ... as $\top \prec\!\!3\, \phi$. Over the classical propositional calculus, the converse holds too ...
- ... i.e., $\phi \prec\!\!3\, \psi$ is same as $\Box(\phi \to \psi)$, i.e., $\top \prec\!\!3\, (\phi \to \psi)$

- As we all know (or do we?) the following is the original syntax of modern modal logic :

$$\mathcal{L}_{⥽} \quad \phi, \psi ::= \top \mid \bot \mid p \mid \phi \to \psi \mid \phi \vee \psi \mid \phi \wedge \psi \mid \phi ⥽ \psi$$

- $⥽$ is the strict implication of Clarence Irving Lewis (1918,1932)

  who is not C.S. Lewis, David Lewis or Lewis Carroll

- $\Box\phi$ is then definable ...
- ... as $\top ⥽ \phi$. Over the classical propositional calculus, the converse holds too ...
- ... i.e., $\phi ⥽ \psi$ is same as $\Box(\phi \to \psi)$, i.e., $\top ⥽ (\phi \to \psi)$
- Truth of strict implication at w = truth of material implication in all possible worlds seen from w

- Lewis indeed wanted to have involutive negation

- Lewis indeed wanted to have involutive negation
- In fact, he introduced $⥽$ as defined using $\Diamond$

  somehow did not explicitly work with $\Box$ in the signature

- Lewis indeed wanted to have involutive negation
- In fact, he introduced $\strictif$ as defined using $\Diamond$

  somehow did not explicitly work with $\Box$ in the signature
- But perhaps this is why $\strictif$ slid into irrelevance ...

- Lewis indeed wanted to have involutive negation
- In fact, he introduced $\prec$ as defined using $\Diamond$

  somehow did not explicitly work with $\Box$ in the signature
- But perhaps this is why $\prec$ slid into irrelevance ...
- ... which did not seem to make him happy

- Lewis indeed wanted to have involutive negation
- In fact, he introduced $\strictif$ as defined using $\Diamond$

  somehow did not explicitly work with $\Box$ in the signature
- But perhaps this is why $\strictif$ slid into irrelevance ...
- ... which did not seem to make him happy
- He didn't even like the name "modal logic" ...

There is a logic restricted to indicatives; the truth-value logic most impressively developed in "Principia Mathematica". But those who adhere to it usually have thought of it—so far as they understood what they were doing—as being the universal logic of propositions which is independent of mode. And when that universal logic was first formulated in exact terms, they failed to recognize it as the only logic which is independent of the mode in which propositions are entertained and dubbed it "modal logic".

- Curiously, Lewis was opened towards non-classical systems (mostly MV of Łukasiewicz)
  - A detailed discussion in *Symbolic Logic*, 1932
  - A paper on "Alternative Systems of Logic", *The Monist*, same year
  - Both references analyze possible definitions of "truth-implications"/"implication-relations" available in finite, but not necessarily binary matrices.
- I found just one reference where he mentions (rather favourably) Brouwer and intuitionism …

[T]he mathematical logician Brouwer has maintained that the law of the Excluded Middle is not a valid principle at all. The issues of so difficult a question could not be discussed here; but let us suggest a point of view at least something like his. . . . The law of the Excluded Middle is not writ in the heavens: it but reflects our rather stubborn adherence to the simplest of all possible modes of division, and our predominant interest in concrete objects as opposed to abstract concepts. The reasons for the choice of our logical categories are not themselves reasons of logic any more than the reasons for choosing Cartesian, as against polar or Gaussian coördinates, are themselves principles of mathematics, or the reason for the radix 10 is of the essence of number.

"Alternative Systems of Logic", *The Monist*, 1932

- No indication he was aware of
- As we will see, maybe he should've followed up on that ...
- ...especially that there were more analogies between him and Brouwer
    - almost perfectly parallel life dates
    - wrote his 1910 PhD on *The Place of Intuition in Knowledge*
    - a solid background in/influence of idealism and Kant ...

New incarnations of strict implication

- Metatheory of arithmetic

  Σ^0_1-preservativity for a theory T extending HA:

  $$A \dashv_T B \Leftrightarrow \forall \Sigma^0_1\text{-sentences } S \; (\; T \vdash S \to A \Rightarrow T \vdash S \to B)$$

  Albert working on this since 1985, later more contributions made also by Iemhoff, de Jongh, Zhou ...

New incarnations of strict implication

- Metatheory of arithmetic

  Σ^0_1-preservativity for a theory T extending HA:

  $$A \dashv_T B \Leftrightarrow \forall \Sigma^0_1\text{-sentences } S \ (\ T \vdash S \to A \Rightarrow T \vdash S \to B)$$

  Albert working on this since 1985, later more contributions made also by Iemhoff, de Jongh, Zhou ...

- Functional programming

  Distinction between arrows of John Hughes and applicative functors/idioms of McBride/Patterson

  A series of papers by Lindley, Wadler, Yallop

New incarnations of strict implication

- Metatheory of arithmetic

  Σ^0_1-preservativity for a theory T extending HA:

  $$A \dashv_T B \Leftrightarrow \forall \Sigma^0_1\text{-sentences } S\ (\ T \vdash S \to A \Rightarrow T \vdash S \to B)$$

  Albert working on this since 1985, later more contributions made also by Iemhoff, de Jongh, Zhou . . .

- Functional programming

  Distinction between arrows of John Hughes and applicative functors/idioms of McBride/Patterson

  A series of papers by Lindley, Wadler, Yallop

- Proof theory of guarded (co)recursion

  Nakano and more recently Clouston&Goré

Idioms are oblivious, arrows are meticulous, monads are promiscuous

Sam Lindley, Philip Wadler and Jeremy Yallop

Laboratory for Foundations of Computer Science
The University of Edinburgh

Abstract

We revisit the connection between three notions of computation: Moggi's *monads*, Hughes's *arrows* and McBride and Paterson's *idioms* (also called *applicative functors*). We show that idioms are equivalent to arrows that satisfy the type isomorphism $A \rightsquigarrow B \simeq 1 \rightsquigarrow (A \to B)$ and that monads are equivalent to arrows that satisfy the type isomorphism $A \rightsquigarrow B \simeq A \to (1 \rightsquigarrow B)$. Further, idioms embed into arrows and arrows embed into monads.

Keywords: applicative functors, idioms, arrows, monads

$$
\begin{tikzcd}
\text{idioms} \arrow[d, bend left] & & \text{monads} \arrow[d, bend left] \\
\begin{array}{c}\text{static arrows}\\ (A \rightsquigarrow B \simeq 1 \rightsquigarrow (A \to B))\end{array} \arrow[u, bend left] \arrow[r, hook] & \text{arrows} \arrow[r, hook] & \begin{array}{c}\text{higher-order arrows}\\ (A \rightsquigarrow B \simeq A \to (1 \rightsquigarrow B))\end{array} \arrow[u, bend left]
\end{tikzcd}
$$

Fig. 1. Idioms, arrows and monads

$\rightsquigarrow$ here is our $\mathrel{-\!\!3}$

ENTCS 2011, proceedings of MSFP 2008

- Each of these motivations could easily fit 30 mins on its own . . .

- Each of these motivations could easily fit 30 mins on its own . . .
- . . . and would interest only a section of the audience

- Each of these motivations could easily fit 30 mins on its own ...
- ...and would interest only a section of the audience
- The body of the work in the metatheory of intuitionistic arithmetic is particularly spectacular ...

- Each of these motivations could easily fit 30 mins on its own …
- …and would interest only a section of the audience
- The body of the work in the metatheory of intuitionistic arithmetic is particularly spectacular …
- …and way too little known

- Each of these motivations could easily fit 30 mins on its own ...
- ...and would interest only a section of the audience
- The body of the work in the metatheory of intuitionistic arithmetic is particularly spectacular ...
- ...and way too little known
- I can only give you a teaser

- Each of these motivations could easily fit 30 mins on its own …
- …and would interest only a section of the audience
- The body of the work in the metatheory of intuitionistic arithmetic is particularly spectacular …
- …and way too little known
- I can only give you a teaser
- …and Kripke semantics is ideal for this

Kripke semantics for intuitionistic $\Box$:

- Nonempty set of worlds
- Two relations:
  - Intuitionistic partial order relation $\preceq$, drawn as $\to$;
  - Modal relation $\sqsubset$, drawn as $\rightsquigarrow$.
- Semantics for $\Box$: $w \Vdash \Box\phi$ if for any $v \sqsupset w$, $v \Vdash \phi$
- Semantics for $-\!\!\!3$:

$$w \Vdash \phi \mathrel{-\!\!\!3} \psi \text{ if for any } v \sqsupset w, v \Vdash \phi \text{ implies } v \Vdash \psi$$

- What it the minimal condition to guarantee persistence?

- What it the minimal condition to guarantee persistence?
- That is, given A, B upward closed, is

$$A \multimap B = \{w \mid \text{ for any } v \sqsupset w, v \in A \text{ implies } v \in B\}$$

upward closed?

- What it the minimal condition to guarantee persistence?
- That is, given A, B upward closed, is

$$A \multimap B = \{w \mid \text{ for any } v \sqsupset w, v \in A \text{ implies } v \in B\}$$

upward closed?
- Is it it stronger than the one ensuring persistence for $\Box A$?

Four frame conditions (known since 1980's)

$$\begin{tikzcd} & \ell \arrow[r, rightsquigarrow] & m \\ k \arrow[ur] \arrow[rr, dashed, rightsquigarrow] & & \ell' \arrow[u, dashed] \end{tikzcd}$$

□-p
(persistence for □)

$$\begin{tikzcd} \ell \arrow[r, rightsquigarrow] & m \\ k \arrow[u] \arrow[ur, dashed, rightsquigarrow] & \end{tikzcd}$$

prefixing
(persistence for ⥽)

$$\begin{tikzcd} & \ell \arrow[dr, rightsquigarrow] & & n \\ k \arrow[ur] \arrow[urrr, dashed, rightsquigarrow] & & m \arrow[ur] & \end{tikzcd}$$

mix /brilliancy

$$\begin{tikzcd} & m \\ k \arrow[r, rightsquigarrow] \arrow[ur, dashed, rightsquigarrow] & \ell \arrow[u] \end{tikzcd}$$

postfixing

⇐ both are equivalent

in presence of □-p
and collapse ⥽ to □

Four frame conditions (known since 1980's)

$$\begin{tikzcd} & \ell \arrow[r, rightsquigarrow] & m \\ k \arrow[ur] \arrow[rr, rightsquigarrow, dashed] & & \ell' \arrow[u, dashed] \end{tikzcd}$$

□-p
(persistence for □)

$$\begin{tikzcd} \ell \arrow[r, rightsquigarrow] & m \\ k \arrow[u] \arrow[ur, rightsquigarrow, dashed] & \end{tikzcd}$$

prefixing
(persistence for ⥽)

$$\begin{tikzcd} & \ell \arrow[dr, rightsquigarrow] & & n \\ k \arrow[ur] \arrow[urrr, rightsquigarrow, dashed] & & m \arrow[ur] & \end{tikzcd}$$

mix /brilliancy

$$\begin{tikzcd} & m \\ k \arrow[r, rightsquigarrow] \arrow[ur, rightsquigarrow, dashed] & \ell \arrow[u] \end{tikzcd}$$

postfixing

$\Leftarrow$ both are equivalent

in presence of □-p
and collapse ⥽ to □

- brilliancy obtains naturally in, e.g., Stone-Jónsson-Tarski for □

Four frame conditions (known since 1980's)

$$\begin{tikzcd} & \ell \arrow[r, squiggly] & m \\ k \arrow[ur] \arrow[rr, squiggly, dashed] & & \ell' \arrow[u, dashed] \end{tikzcd}$$

$\Box$-p
(persistence for $\Box$)

$$\begin{tikzcd} \ell \arrow[r, squiggly] & m \\ k \arrow[u] \arrow[ur, squiggly, dashed] & \end{tikzcd}$$

prefixing
(persistence for $\strictif$)

$$\begin{tikzcd} & \ell \arrow[dr, squiggly] & & n \\ k \arrow[ur] \arrow[urrr, squiggly, dashed] & & m \arrow[ur] & \end{tikzcd}$$

mix /brilliancy

$$\begin{tikzcd} & m \\ k \arrow[r, squiggly] \arrow[ur, squiggly, dashed] & \ell \arrow[u] \end{tikzcd}$$

postfixing

$\Leftarrow$ both are equivalent

in presence of $\Box$-p
and collapse $\strictif$ to $\Box$

- brilliancy obtains naturally in, e.g., Stone-Jónsson-Tarski for $\Box$
- ...but $\strictif$ can feel it!

Four frame conditions (known since 1980's)

$$\begin{tikzcd} & \ell \arrow[r, rightsquigarrow] & m \\ k \arrow[ur] \arrow[rr, rightsquigarrow, dashed] & & \ell' \arrow[u, dashed] \end{tikzcd}$$

$\Box$-p

(persistence for $\Box$)

$$\begin{tikzcd} \ell \arrow[r, rightsquigarrow] & m \\ k \arrow[u] \arrow[ur, rightsquigarrow, dashed] & \end{tikzcd}$$

prefixing

(persistence for $\strictif$)

$$\begin{tikzcd} & \ell \arrow[dr, rightsquigarrow] & & n \\ k \arrow[ur] \arrow[urrr, rightsquigarrow, dashed] & & m \arrow[ur] & \end{tikzcd}$$

mix / brilliancy

$$\begin{tikzcd} & m \\ k \arrow[r, rightsquigarrow] \arrow[ur, rightsquigarrow, dashed] & \ell \arrow[u] \end{tikzcd}$$

postfixing

$\Leftarrow$ both are equivalent

in presence of $\Box$-p
and collapse $\strictif$ to $\Box$

- brilliancy obtains naturally in, e.g., Stone-Jónsson-Tarski for $\Box$
- ... but $\strictif$ can feel it!
- It is precisely the condition ensuring collapse of $\strictif$ to $\Box$

Four frame conditions (known since 1980's)

$$\begin{tikzcd}
& \ell \arrow[r, rightsquigarrow] & m \\
k \arrow[ur] \arrow[rr, dashed, rightsquigarrow] & & \ell' \arrow[u, dashed]
\end{tikzcd}$$

$\Box$-p
(persistence for $\Box$)

$$\begin{tikzcd}
\ell \arrow[r, rightsquigarrow] & m \\
k \arrow[u] \arrow[ur, dashed, rightsquigarrow] &
\end{tikzcd}$$

prefixing
(persistence for $\strictif$)

$$\begin{tikzcd}
& \ell \arrow[dr, rightsquigarrow] & & n \\
k \arrow[ur] \arrow[urrr, dashed, rightsquigarrow] & & m \arrow[ur] &
\end{tikzcd}$$

mix /brilliancy

$$\begin{tikzcd}
& m \\
k \arrow[r, rightsquigarrow] \arrow[ur, dashed, rightsquigarrow] & \ell \arrow[u]
\end{tikzcd}$$

postfixing

$\Leftarrow$ both are equivalent

in presence of $\Box$-p
and collapse $\strictif$ to $\Box$

- brilliancy obtains naturally in, e.g., Stone-Jónsson-Tarski for $\Box$
- ...but $\strictif$ can feel it!
- It is precisely the condition ensuring collapse of $\strictif$ to $\Box$
- Over prefixing (or $\strictif$-frames) $\Box(\phi \to \psi)$ implies $\phi \strictif \psi$, but not the other way around

Idioms are oblivious, arrows are meticulous, monads are promiscuous

Sam Lindley, Philip Wadler and Jeremy Yallop

Laboratory for Foundations of Computer Science
The University of Edinburgh

Abstract

We revisit the connection between three notions of computation: Moggi's *monads*, Hughes's *arrows* and McBride and Paterson's *idioms* (also called *applicative functors*). We show that idioms are equivalent to arrows that satisfy the type isomorphism $A \leadsto B \simeq 1 \leadsto (A \to B)$ and that monads are equivalent to arrows that satisfy the type isomorphism $A \leadsto B \simeq A \to (1 \leadsto B)$. Further, idioms embed into arrows and arrows embed into monads.

Keywords: applicative functors, idioms, arrows, monads

$$
\begin{tikzcd}
\text{idioms} \arrow[d, bend left] & & \text{monads} \arrow[d, bend left] \\
\text{static arrows} \arrow[u, bend left] \arrow[r, hook] & \text{arrows} \arrow[r, hook] & \text{higher-order arrows} \arrow[u, bend left]
\end{tikzcd}
$$

$(A \leadsto B \simeq 1 \leadsto (A \to B))$ $\qquad\qquad$ $(A \leadsto B \simeq A \to (1 \leadsto B))$

Fig. 1. Idioms, arrows and monads

$\leadsto$ here is our $\multimap$

ENTCS 2011, proceedings of MSFP 2008

- I'd suggest calling FP arrows "strong arrows"

- I'd suggest calling FP arrows "strong arrows"
- They satisfy in addition the axiom $(\phi \to \psi) \to \phi \strictif \psi$

- I’d suggest calling FP arrows “strong arrows”
- They satisfy in addition the axiom $(\phi \to \psi) \to \phi \strictif \psi$
- …or, equivalently, $\mathsf{S_a} \quad \phi \to \Box\phi$

- I'd suggest calling FP arrows "strong arrows"
- They satisfy in addition the axiom $(\phi \to \psi) \to \phi \strictif \psi$
- …or, equivalently, $\mathsf{S_a} \quad \phi \to \Box\phi$
- Why "equivalently"?

$$\begin{aligned} \phi \to \psi &\leq \Box(\phi \to \psi) \\ &\leq \phi \strictif \psi \end{aligned}$$

- I'd suggest calling FP arrows "strong arrows"
- They satisfy in addition the axiom $(\phi \to \psi) \to \phi \multimap \psi$
- …or, equivalently, $\mathsf{S_a} \quad \phi \to \Box\phi$
- Why "equivalently"?

$$\begin{aligned} \phi \to \psi &\leq \Box(\phi \to \psi) \\ &\leq \phi \multimap \psi \end{aligned}$$

- This forces $\sqsubset$ to be contained in $\preceq$

- I'd suggest calling FP arrows "strong arrows"
- They satisfy in addition the axiom $(\phi \to \psi) \to \phi \mathrel{-\!\!3} \psi$
- …or, equivalently, $\mathsf{S_a} \quad \phi \to \Box\phi$
- Why "equivalently"?

$$\begin{aligned} \phi \to \psi &\leq \Box(\phi \to \psi) \\ &\leq \phi \mathrel{-\!\!3} \psi \end{aligned}$$

- This forces $\sqsubset$ to be contained in $\preceq$
- …rather degenerate in the classical case …

- I'd suggest calling FP arrows "strong arrows"
- They satisfy in addition the axiom $(\phi \to \psi) \to \phi \mathrel{-\!\!3} \psi$
- ...or, equivalently, $\mathsf{S_a}$ $\quad \phi \to \Box\phi$
- Why "equivalently"?

$$\begin{aligned} \phi \to \psi &\leq \Box(\phi \to \psi) \\ &\leq \phi \mathrel{-\!\!3} \psi \end{aligned}$$

- This forces $\sqsubset$ to be contained in $\preceq$
- ...rather degenerate in the classical case ...
- ...only three consistent logics of (disjoint unions of) singleton(s) ...

- I'd suggest calling FP arrows "strong arrows"
- They satisfy in addition the axiom $(\phi \to \psi) \to \phi \multimap \psi$
- …or, equivalently, $\mathsf{S_a} \quad \phi \to \Box\phi$
- Why "equivalently"?

$$\begin{aligned} \phi \to \psi &\leq \Box(\phi \to \psi) \\ &\leq \phi \multimap \psi \end{aligned}$$

- This forces $\sqsubset$ to be contained in $\preceq$
- …rather degenerate in the classical case …
- …only three consistent logics of (disjoint unions of) singleton(s) …
- …and yet intuitionistically you have a whole zoo: logics of (type inhabitation of) idioms, arrows, strong monads/PLL with superintuitionistic logics as a degenerate case

  also recent attempts at "intuitionistic epistemic logics", esp. Artemov and Protopopescu

Axioms and rules of iA^-:

Those of IPC plus:

$$\mathsf{Tra} \quad \phi \mathrel{-\!\!3} \psi \to \psi \mathrel{-\!\!3} \chi \to \phi \mathrel{-\!\!3} \chi$$

$$\mathsf{K_a} \quad \phi \mathrel{-\!\!3} \psi \to \phi \mathrel{-\!\!3} \chi \to \phi \mathrel{-\!\!3} (\psi \wedge \chi)$$

$$\mathsf{N_a} \ \frac{\phi \to \psi}{\phi \mathrel{-\!\!3} \psi.}$$

Axioms and rules of iA^-:

Those of IPC plus:

$$\mathsf{Tra} \quad \phi \dashv \psi \to \psi \dashv \chi \to \phi \dashv \chi$$

$$\mathsf{K_a} \quad \phi \dashv \psi \to \phi \dashv \chi \to \phi \dashv (\psi \wedge \chi)$$

$$\mathsf{N_a} \ \frac{\phi \to \psi}{\phi \dashv \psi.}$$

Axioms and rules of the full minimal system iA:

All the axioms and rules of IPC and iA^- and

$$\mathsf{Di} \quad \phi \dashv \chi \to \psi \dashv \chi \to (\phi \vee \psi) \dashv \chi.$$

Derivation exercises

Lots to be found in our paper, e.g., a generalization of $\mathsf{K_a}$:

$$\begin{aligned} \phi \mathrel{-\!\!\!\!\rightarrow} (\psi \to \chi) &\vdash (\phi \wedge \psi) \mathrel{-\!\!\!\!\rightarrow} (\psi \wedge (\psi \to \chi)) && \text{by } \mathsf{N_a} \text{ and } \mathsf{K_a} \\ &\vdash (\phi \wedge \psi) \mathrel{-\!\!\!\!\rightarrow} \chi && \text{by monotonicity of } \mathrel{-\!\!\!\!\rightarrow} \end{aligned}$$

Derivation exercises

Lots to be found in our paper, e.g., a generalization of $\mathsf{K_a}$:

$$\phi \multimapinv (\psi \to \chi) \vdash (\phi \wedge \psi) \multimapinv (\psi \wedge (\psi \to \chi)) \qquad \text{by } \mathsf{N_a} \text{ and } \mathsf{K_a}$$

$$\vdash (\phi \wedge \psi) \multimapinv \chi \qquad \text{by monotonicity of } \multimapinv$$

Another curious one:

$$\psi \multimapinv \chi \vdash \psi \multimapinv (\psi \to \chi) \wedge \neg\psi \multimapinv (\psi \to \chi) \qquad \text{by } \mathsf{Tra} \text{ and } \mathsf{N_a}$$

$$\vdash (\psi \vee \neg\psi) \multimapinv (\psi \to \chi) \qquad \text{by } \mathsf{Di}$$

Derivation exercises

Lots to be found in our paper, e.g., a generalization of $\mathsf{K_a}$:

$$\begin{aligned} \phi \mathrel{-\!\!3} (\psi \to \chi) &\vdash (\phi \wedge \psi) \mathrel{-\!\!3} (\psi \wedge (\psi \to \chi)) && \text{by } \mathsf{N_a} \text{ and } \mathsf{K_a} \\ &\vdash (\phi \wedge \psi) \mathrel{-\!\!3} \chi && \text{by monotonicity of } \mathrel{-\!\!3} \end{aligned}$$

Another curious one:

$$\begin{aligned} \psi \mathrel{-\!\!3} \chi &\vdash \psi \mathrel{-\!\!3} (\psi \to \chi) \wedge \neg\psi \mathrel{-\!\!3} (\psi \to \chi) && \text{by } \mathsf{Tra} \text{ and } \mathsf{N_a} \\ &\vdash (\psi \vee \neg\psi) \mathrel{-\!\!3} (\psi \to \chi) && \text{by } \mathsf{Di} \end{aligned}$$

We thus get

$$\psi \mathrel{-\!\!3} \chi \dashv\vdash (\psi \vee \neg\psi) \mathrel{-\!\!3} (\psi \to \chi)$$

- The validity of

$$p \mathrel{-\!\!3} q \dashv\vdash (p \vee \neg p) \mathrel{-\!\!3} (p \to q)$$

implies that CoI is valid over classical logic

- The validity of

$$p \multimap q \dashv\vdash (p \lor \neg p) \multimap (p \to q)$$

  implies that CoI is valid over classical logic
- We derived syntactically why you need IPC to get $\multimap$ to work

- The validity of

$$p \strictif q \dashv\vdash (p \vee \neg p) \strictif (p \to q)$$

  implies that CoI is valid over classical logic
- We derived syntactically why you need IPC to get $\strictif$ to work
- Note no other classical tautology in one variable would do:

$$p \strictif q \nvdash (\neg\neg p \to p) \strictif (p \to q)$$

- Completeness results for many such systems published by Iemhoff et al

  Her 2001 PhD, 2003 MLQ, 2005 SL with de Jongh and Zhou

  Also Zhou's ILLC MSc in 2003

- Completeness results for many such systems published by Iemhoff et al

  Her 2001 PhD, 2003 MLQ, 2005 SL with de Jongh and Zhou

  Also Zhou's ILLC MSc in 2003

- In our paper, we announce more such completeness and correspondence results

  based on on a suitable extension of Gödel-McKinsey-Tarski and

  Wolter-Zakharyaschev for ordinary intuitionistic modal logics

  Details to be published separately

- Finally, a few words on preservativity

- Finally, a few words on preservativity
- Let us first recall the simpler idea of the logic of *provability* ...

- Finally, a few words on preservativity
- Let us first recall the simpler idea of the logic of *provability* ...
- ... or even more generally, that of arithmetical interpretation of a propositional logic

- Extend $\mathcal{L}$ to $\mathcal{L}_{\circledcirc_0,\ldots,\circledcirc_k}$. with operators $\circledcirc_0, \ldots, \circledcirc_k$

  where $\circledcirc_i$ has arity n_i

- F assigns to every $\circledcirc_i$ an arithmetical formula $A(v_0, \ldots, v_{n_i-1})$

  where all free variables are among the variables shown

- We write $\circledcirc_{i,F}(B_0, \ldots, B_{n_i-1})$ for $F(\circledcirc_i)(\underline{\ulcorner B_0 \urcorner}, \ldots, \underline{\ulcorner B_{n_i-1} \urcorner})$

  Here $\underline{\ulcorner C \urcorner}$ is the numeral of the Gödel number of C

- f maps $Vars$ to arithmetical sentences. Define $(\phi)^f_F$:
  - $(p)^f_F := f(p)$
  - $(\cdot)^f_F$ commutes with the propositional connectives
  - $(\circledcirc_i(\phi_0, \ldots, \phi_{n_i-1}))^f_F := \circledcirc_F((\phi_0)^f_F, \ldots, (\phi_{n_i-1})^f_F)$

- Let T be an arithmetical theory

  An extension of i-EA, the intuitionistic version of Elementary Arithmetic, in the arithmetical language

- A modal formula in $\mathcal{L}_{\circledcirc_0,\ldots,\circledcirc_k}$ is T-valid w.r.t. F iff, for all assignments f of arithmetical sentences to $Vars$, we have $T \vdash (\phi)^f_F$.
- Write $\Lambda_{T,F}$ for the set of $\mathcal{L}_{\circledcirc_0,\ldots,\circledcirc_k}$-formulas that are T-valid w.r.t. F.
- Of course, $\Lambda_{T,F}$ interesting only for well-chosen F

- First, consider a single unary $\circledcirc = \Box$ and any arithmetical theory T ...
- ... which comes equipped with a $\Delta_0(\mathsf{exp})$-predicate α_T encoding its axiom set.
- Let provability in T be arithmetised by prov_T.
- Set $\mathsf{F}_{0,T}(\Box) := \mathsf{prov}_\mathsf{T}(v_0)$. Let $\Lambda^*_T := \Lambda_{T,\mathsf{F}_{0,T}}$.
- Intuitionistic Löb's logic $\mathsf{i\text{-}GL}$ is given by the following axioms over IPC.

$$\begin{array}{ll} \mathsf{N} & \vdash \phi \;\Rightarrow \vdash \Box\phi \\ \mathsf{K} & \vdash \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi) \\ \mathsf{L} & \vdash \Box(\Box\phi \to \phi) \to \Box\phi \end{array}$$

The theory GL is obtained by extending $\mathsf{i\text{-}GL}$ with classical logic

If T is a Σ^1_0-sound classical theory, then $\Lambda^*_T = \mathsf{GL}$ (Solovay)

In contrast, the logic $\mathsf{i\text{-}GL}$ is not complete for HA:

- $\vdash \Box\neg\neg\Box\phi \to \Box\Box\phi$.
- $\vdash \Box(\neg\neg\Box\phi \to \Box\phi) \to \Box\Box\phi$
- $\vdash \Box(\phi \vee \psi) \to \Box(\phi \vee \Box\psi)$.

Still unknown what the ultimate axiomatization is

- Many possible interpretations of a binary connective

  not all of them producing Lewis' arrows!

- Interpretability
- Π^0_1-conservativity
- Σ^0_1-preservativity

  classically, the last two intertranslatable, like $\Box$ and $\Diamond$

- The notion of Σ_1^0-preservativity for a theory T (Visser 1985) is defined as follows:
- $A \prec_T B$ iff, for all Σ_1^0-sentences S, if $T \vdash S \to A$, then $T \vdash S \to B$
- This does yields Lewis' arrow …
- …with interesting additional axioms

Examples of valid principles

4_a $\vdash \phi \dashv\!\!\!\!\!\!\exists \Box\phi$

L_a $(\Box\phi \to \phi) \dashv\!\!\!\!\!\!\exists \phi$

W_a $(\phi \land \Box\psi) \dashv\!\!\!\!\!\!\exists \psi \to \phi \dashv\!\!\!\!\!\!\exists \psi$

W'_a $\phi \dashv\!\!\!\!\!\!\exists \psi \to (\Box\psi \to \phi) \dashv\!\!\!\!\!\!\exists \psi$

M_a $\phi \dashv\!\!\!\!\!\!\exists \psi \to (\Box\chi \to \phi) \dashv\!\!\!\!\!\!\exists (\Box\chi \to \psi)$

M'_a $(\phi \land \Box\chi) \dashv\!\!\!\!\!\!\exists \psi \to \phi \dashv\!\!\!\!\!\!\exists (\Box\chi \to \psi)$

- Still no ultimate axiomatization... but perhaps better candidates and better insights than for $\Box$ only, see our paper

Additional axioms in well-behaved/pathological theories

E.g., in presence of The Completeness Principle for a theory T:

S_a $(\phi \to \psi) \to \phi \dashv\!\!\!\!\!\!\exists \psi$, i.e., S'_a: $\phi \dashv\!\!\!\!\!\!\exists \psi \to \phi \to \Box\psi$

- Our present work includes computation of fixpoints of modalized formulas
- (below Sa, is more interesting than in presence of $\Box$ only!)
- ...encoding of fixpoints of positive formulas and retraction of μ-calculus

- Happy birthday to you, Ishihara-sensei!
- A word from Albert:
  A very nice program and some well-known speakers.
- He asked me to pass his greetings to numerous friends here