Normal modal logics and provability predicates

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Outline

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1. Provability predicates
2. Arithmetical interpretations and provability logics
3. Our results

Provability predicates

- $\mathcal{L}_A$: the language of first-order arithmetic
- $\overline{n}$: the numeral for $n \in \omega$

In the usual proof of Gödel's incompleteness theorems, a provability predicate plays an important role.

Provability predicates

A formula $\mathsf{Pr}(x)$ is a provability predicate of PA
$\stackrel{\text{def.}}{\iff}$ for any $n \in \omega$,
$\mathsf{PA} \vdash \mathsf{Pr}(\overline{n}) \iff n$ is the Gödel number of some theorem of PA.

Standard construction of provability predicates

Gödel-Feferman's standard construction of provability predicates of PA is as follows.

Numerations

A formula $\tau(v)$ is a numeration of PA
$\overset{\text{def.}}{\Longleftrightarrow}$ **for any $n \in \omega$,**
$\mathsf{PA} \vdash \tau(\overline{n}) \iff n$ **is the Gödel number of an axiom of PA.**

- **Let $\tau(v)$ be a numeration of PA.**
- **The relation "x is the Gödel number of an $\mathcal{L}_A$-fomula provable in the theory defined by $\tau(v)$" is naturally expressed in the language $\mathcal{L}_A$.**
- **The resulting $\mathcal{L}_A$-formula is denoted by $\mathsf{Pr}_\tau(x)$.**
- **If $\tau(v)$ is Σ_{n+1}, then $\mathsf{Pr}_\tau(x)$ is also Σ_{n+1}.**

Properties of standard provability predicates

Theorem (Hilbert-Bernays-Löb-Feferman)

Let $\tau(v)$ be any numeration of PA.

- $\mathsf{Pr}_\tau(x)$ **is a provability predicate of PA.**
- $\mathsf{PA} \vdash \mathsf{Pr}_\tau(\ulcorner \varphi \to \psi \urcorner) \to (\mathsf{Pr}_\tau(\ulcorner \varphi \urcorner) \to \mathsf{Pr}_\tau(\ulcorner \psi \urcorner))$.
- $\mathsf{PA} \vdash \varphi \to \mathsf{Pr}_\tau(\ulcorner \varphi \urcorner)$ **for any Σ_1 sentence φ.**

Theorem

Let $\tau(v)$ be any Σ_1 numeration of PA.

- $\mathsf{PA} \vdash \mathsf{Pr}_\tau(\ulcorner \varphi \urcorner) \to \mathsf{Pr}_\tau(\ulcorner \mathsf{Pr}_\tau(\ulcorner \varphi \urcorner) \urcorner)$.
- **(Gödel's second incompleteness theorem)**
  $\mathsf{PA} \nvdash \mathsf{Con}_\tau$,
  where Con_τ is the consistency statement $\neg \mathsf{Pr}_\tau(\ulcorner \overline{0} = \overline{1} \urcorner)$ of $\tau(v)$.
- **(Löb's theorem)**
  $\mathsf{PA} \vdash \mathsf{Pr}_\tau(\ulcorner \mathsf{Pr}_\tau(\ulcorner \varphi \urcorner) \to \varphi \urcorner) \to \mathsf{Pr}_\tau(\ulcorner \varphi \urcorner)$.

Nonstandard provability predicates

There are many nonstandard provability predicates.

- **Rosser's provability predicate**
  $\mathsf{Pr}^R(x) \equiv \exists y(\mathsf{Prf}(x, y) \land \forall z \leq y \neg \mathsf{Prf}(\dot{\neg} x, z))$,
  where $\mathsf{Prf}(x, y)$ **is a** Δ_1 **proof predicate.**
- **Mostowski's provability predicate**
  $\mathsf{Pr}^M(x) \equiv \exists y(\mathsf{Prf}(x, y) \land \neg \mathsf{Prf}(\ulcorner \overline{0} = \overline{1} \urcorner, y))$
- **Shavrukov's provability predicate**
  $\mathsf{Pr}^S(x) \equiv \exists y(\mathsf{Pr}_{I\Sigma_y}(x) \land \mathsf{Con}_{I\Sigma_y})$
- $\cdots$

Problem

What are the PA-provable principles of each provability predicate?

This problem is investigated in the framework of modal logic.

Outline

Modal logics

Axioms and Rules of the modal logic K

Axioms Tautologies and $\Box(p \to q) \to (\Box p \to \Box q)$.

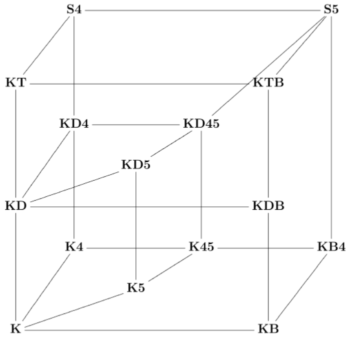
Rules Modus ponens $\dfrac{\varphi, \ \varphi \to \psi}{\psi}$, Necessitation $\dfrac{\varphi}{\Box\varphi}$, and Substitution.

Normal modal logics

A modal logic L is **normal**
$\overset{\text{def.}}{\iff}$ L includes K and is closed under three rules of K.

For each modal formula A, $L + A$ denotes the smallest normal modal logic including L and A.

- $\mathsf{KT} = \mathsf{K} + \Box p \to p$
- $\mathsf{KD} = \mathsf{K} + \neg\Box\bot$
- $\mathsf{K4} = \mathsf{K} + \Box p \to \Box\Box p$
- $\mathsf{K5} = \mathsf{K} + \Diamond p \to \Box\Diamond p$
- $\mathsf{KB} = \mathsf{K} + p \to \Box\Diamond p$
- $\mathsf{GL} = \mathsf{K} + \Box(\Box p \to p) \to \Box p$
- $\cdots$



Arithmetical interpretations and provability logics

Let $\mathsf{Pr}(x)$ be a provability predicate of PA.

Arithmetical interpretations

A mapping f from modal formulas to $\mathcal{L}_A$-sentences is an arithmetical interpretation based on $\mathsf{Pr}(x)$
$\overset{\text{def.}}{\iff}$ **f satisfies the following conditions:**

- $f(\bot) \equiv \overline{0} = \overline{1}$;
- $f(A \to B) \equiv f(A) \to f(B)$;
- $\cdots$
- $f(\Box A) \equiv \mathsf{Pr}(\ulcorner f(A) \urcorner)$.

Provability logics

$\mathsf{PL}(\mathsf{Pr}) := \{A : \mathsf{PA} \vdash f(A)$ **for all arithmetical interpretations f based on** $\mathsf{Pr}(x)\}$.
The set $\mathsf{PL}(\mathsf{Pr})$ is said to be the provability logic of $\mathsf{Pr}(x)$.

Solovay's arithmetical completeness theorem

- **Recall that for each Σ_1 numeration $\tau(v)$ of PA,**
  - $\mathsf{PA} \vdash \mathsf{Pr}_\tau(\ulcorner \varphi \to \psi \urcorner) \to (\mathsf{Pr}_\tau(\ulcorner \varphi \urcorner) \to \mathsf{Pr}_\tau(\ulcorner \psi \urcorner))$,
  - $\mathsf{PA} \vdash \mathsf{Pr}_\tau(\ulcorner \mathsf{Pr}_\tau(\ulcorner \varphi \urcorner) \to \varphi \urcorner) \to \mathsf{Pr}_\tau(\ulcorner \varphi \urcorner)$.
- **Corresponding modal formulas $\Box(p \to q) \to (\Box p \to \Box q)$ and $\Box(\Box p \to p) \to \Box p$ are axioms of GL.**
- **In fact, GL is exactly the provability logic of standard Σ_1 provability predicates.**

Arithmetical completeness theorem (Solovay, 1976)

For any Σ_1 numeration $\tau(v)$ of PA, $\mathsf{PL}(\mathsf{Pr}_\tau)$ coincides with GL.

Feferman's predicate

On the other hand, there are provability predicates whose provability logics are completely different from GL.

Theorem (Feferman, 1960)

There exists a Π_1 numeration $\tau(v)$ of PA such that $\mathsf{PA} \vdash \mathsf{Con}_\tau$.
Consequently, $\mathsf{KD} \subseteq \mathsf{PL}(\mathsf{Pr}_\tau)$ ($\mathsf{KD} = \mathsf{K} + \neg\Box\bot$).

Shavrukov found a nonstandard provability predicate whose provability logic is strictly stronger than KD.

Theorem (Shavrukov, 1994)

Let $\mathsf{Pr}^S(x) \equiv \exists y(\mathsf{Pr}_{I\Sigma_y}(x) \land \mathsf{Con}_{I\Sigma_y})$.
Then $\mathsf{PL}(\mathsf{Pr}^S) = \mathsf{KD} + \Box p \to \Box((\Box q \to q) \lor \Box p)$.

- **There may be a lot of normal modal logic which is the provability logic of some provability predicate.**
- **We are interested in the following general problem.**

General Problem

Which normal modal logic is the provability logic $\mathsf{PL}(\mathsf{Pr})$ of some provability predicate $\mathsf{Pr}(x)$ of PA?

- **Kurahashi, T., Arithmetical completeness theorem for modal logic** K, *Studia Logica*, **to appear.**
- **Kurahashi, T., Arithmetical soundness and completeness for Σ_2 numerations,** *Studia Logica*, **to appear.**
- **Kurahashi, T., Rosser provability and normal modal logics, submitted.**

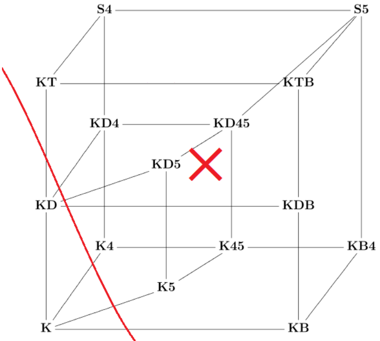
Outline

Several normal modal logics cannot be of the form PL(Pr).

Proposition (K., 201x)

Let L be a normal modal logic satisfying one of the following conditions. Then $L \neq \mathsf{PL}(\mathsf{Pr})$ for all provability predicates $\mathsf{Pr}(x)$ of PA.

1. $\mathsf{KT} \subseteq L$.
2. $\mathsf{K4} \subseteq L$ and $\mathsf{GL} \not\subseteq L$.
3. $\mathsf{K5} \subseteq L$.
4. $\mathsf{KB} \subseteq L$.



Theorem 1

There exists a numeration of PA whose provability logic is minimum.

Theorem 1 (K., 201x)

There exists a Σ_2 numeration $\tau(v)$ of PA such that $\mathsf{PL}(\mathsf{Pr}_\tau) = \mathsf{K}$.

Theorem 2

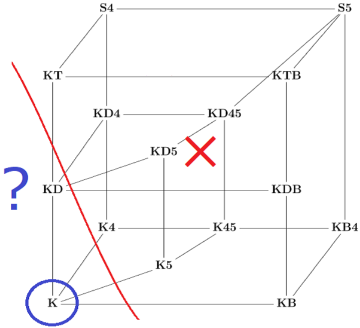
- **Sacchetti (2001) introduced the logics** $\mathsf{K} + \Box(\Box^n p \to p) \to \Box p$ **(**$n \geq 2$**).**
- **For** $n \geq 2$**,** $\mathsf{K} + \Box(\Box^n p \to p) \to \Box p \subsetneq \mathsf{GL}$**.**
- **He conjectured that these logics are provability logics of some nonstandard provability predicates.**

We gave a proof of this conjecture.

Theorem 2 (K., 201x)

For each $n \geq 2$**, there exists a** Σ_2 **numeration** $\tau(v)$ **of PA such that** $\mathsf{PL}(\mathsf{Pr}_\tau) = \mathsf{K} + \Box(\Box^n p \to p) \to \Box p$**.**

Therefore there are infinitely many provability logics.



- **How about** KD**?**
- **We paid attention to Rosser's provability predicates** $\mathsf{Pr}^R(x)$ **because** PA **always proves the consistency statements** Con^R **defined by using** $\mathsf{Pr}^R(x)$**.**

Rosser's provability predicates

However, provability logics of Rosser's provability predicates are sometimes not normal.

Theorem (Guaspari and Solovay, 1979)

There exists a Rosser provability predicate $\mathsf{Pr}^R(x)$ such that $\mathsf{PA} \nvdash \mathsf{Pr}^R(\ulcorner \varphi \to \psi \urcorner) \to (\mathsf{Pr}^R(\ulcorner \varphi \urcorner) \to \mathsf{Pr}^R(\ulcorner \psi \urcorner))$ for some φ and ψ.

On the other hand, there exists a Rosser provability predicate whose provability logic is normal.

Theorem (Arai, 1990)

There exists a Rosser provability predicate $\mathsf{Pr}^R(x)$ such that $\mathsf{PA} \vdash \mathsf{Pr}^R(\ulcorner \varphi \to \psi \urcorner) \to (\mathsf{Pr}^R(\ulcorner \varphi \urcorner) \to \mathsf{Pr}^R(\ulcorner \psi \urcorner))$ for any φ and ψ.

Then $\mathsf{KD} \subseteq \mathsf{PL}(\mathsf{Pr}^R)$ for Arai's predicate $\mathsf{Pr}^R(x)$.

Theorems 3 and 4

We proved that there exists $\mathsf{Pr}^R(x)$ whose provability logic coincides with KD.

Theorem 3 (K., 201x)

There exists a Rosser provability predicate $\mathsf{Pr}^R(x)$ such that $\mathsf{PL}(\mathsf{Pr}^R) = \mathsf{KD}$.

Moreover, there exists a Rosser provability predicate whose provability logic is strictly stronger than KD.

Theorem 4 (K., 201x)

There exists a Rosser provability predicate $\mathsf{Pr}^R(x)$ such that $\mathsf{KD} + \Box\neg p \to \Box\neg\Box p \subseteq \mathsf{PL}(\mathsf{Pr}^R)$.

Open Problems

Open Problem 1

Are there any others logics L such that $\mathsf{K} \subsetneq L \subsetneq \mathsf{GL}$ and $L = \mathsf{PL}(\mathsf{Pr})$ for some $\mathsf{Pr}(x)$?

Open Problem 2

Is there a numeration $\tau(v)$ of PA such that $\mathsf{PL}(\mathsf{Pr}_\tau) = \mathsf{KD}$?

Open Problem 3

Is there a Rosser provability predicate $\mathsf{Pr}^R(x)$ such that $\mathsf{PL}(\mathsf{Pr}^R) = \mathsf{KD} + \Box\neg p \to \Box\neg\Box p$?

General Problem

Which (normal) modal logic is in the set $\{\mathsf{PL}(\mathsf{Pr}) : \mathsf{Pr}(x)$ is a provability predicate of PA$\}$?