Ordinal Ranks on the Baire and non-Baire class functions

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There are various notions of reducibility for decision problems;
e.g. many-one (***m***), truth-table (***tt***), and Turing (***T***) reducibility

- Day-Downey-Westrick (DDW) recently introduced ***m***-, ***tt***-, and ***T***-reducibility for real-valued functions.
- We give a full description of the structures of DDW's ***m***- and ***T***-degrees of real-valued functions.

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- Day-Downey-Westrick (DDW) recently introduced ***m***-, ***tt***-, and ***T***-reducibility for real-valued functions.
- We give a full description of the structures of DDW's ***m***- and ***T***-degrees of real-valued functions.

Caution: Without mentioning, we always assume **AD** (axiom of determinacy).

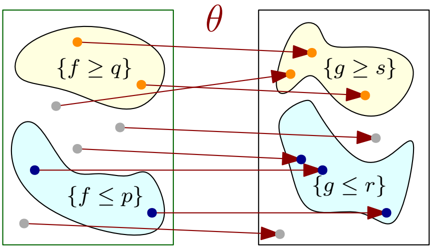
But, of course:

- If we restrict our attention to Borel sets and Baire functions, every result presented in this talk is provable within ZFC.
- If we restrict our attention to projective sets and functions, every result presented in this talk is provable within ZF+DC+PD.
- We even have $L(\mathbb{R}) \models \mathbf{AD}^+$, assuming that there are arbitrarily large Woodin cardinals.

Day-Downey-Westrick's m-reducibility

For $f, g : 2^\omega \to \mathbb{R}$, say f is m-reducible to g (written $f \leq_m g$) if given $p < q$, there are $r < s$ and continuous $\theta : 2^\omega \to 2^\omega$ s.t.

- If $f(x) \leq p$ then $g(\theta(x)) \leq r$.
- If $f(x) \geq q$ then $g(\theta(x)) \geq s$.



Definition (Bourgain 1980)

Let $f : X \to \mathbb{R}$, $p < q$, and $S \subseteq X$.
Define the (f, p, q)-derivative $D_{f,p,q}(S)$ of S as follows.

$$S \setminus \bigcup \{x \in S : (\exists U \ni x)\ f[U] \subseteq (-\infty, q) \text{ or } f[U] \subseteq (p, \infty)\},$$

where U ranges over open sets. Consider the derivation procedure

$$P^0_{f,p,q} = X,\ P^{\xi+1}_{f,p,q} = D_{f,p,q}(P^\xi_{f,p,q}),\ P^\lambda_{f,p,q} = \bigcap_{\xi<\lambda} P^\xi_{f,p,q} \text{ for } \lambda \text{ limit}$$

The Bourgain rank $\alpha(f)$ is defined as follows:

$$\alpha(f, p, q) = \min\{\alpha : P^\alpha_{f,p,q} = \emptyset\}; \quad \alpha(f) = \sup_{p<q} \alpha(f, p, q).$$

- $\alpha(f) = 1 \iff f$ is continuous.
- The rank $\alpha(f)$ exists $\iff f$ is a Baire-one function.
  (f is Baire-one iff it is a pointwise limit of continuous functions)

(Bourgain introduced this notion to analyze the Odell-Rosenthal theorem in Banach space theory)

Definition (Day-Downey-Westrick)

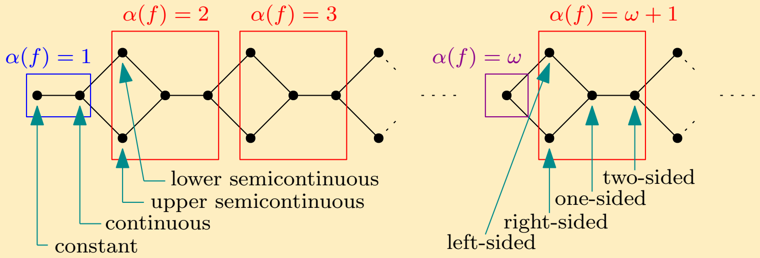
Let $f : X \to \mathbb{R}$ be a Baire-one function.

1. f is *two-sided* if $\exists p < q$ s.t. $\alpha(f, p, q) = \alpha(f)$ and $\forall \nu < \alpha(f) \exists x, y$ rank $> \nu$ s.t. $f(x) \leq p < q \leq f(y)$.
2. If f is not two-sided, it is called *one-sided*.
3. f is *left-sided* if $\forall p < q$ with $\alpha(f, p, q) = \alpha(f)$, $\exists \nu < \alpha(f) \forall x$ rank $> \nu$, $f(x) < p$.
4. f is *right-sided* if $\forall p < q$ with $\alpha(f, p, q) = \alpha(f)$, $\exists \nu < \alpha(f) \forall x$ rank $> \nu$, $f(x) > q$.
5. f is *free one-sided* if it is one-, but neither left- nor right-sided.

- Rank **2** and left-sided $\iff$ lower semicontinuous
- Rank **2** and right-sided $\iff$ upper semicontinuous

Theorem (Day-Downey-Westrick)

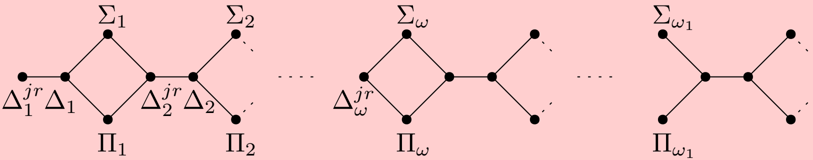
- For Baire-one functions, $\alpha(f) \leq \alpha(g)$ implies $f \leq_m g$.
- The α-rank **1** consists of two m-degrees.
- Each successor α-rank > 1 consists of four m-degrees.
- Each limit α-rank consists of a single m-degree.



This gives a full description of the m-degrees of the Baire-one functions.

1st Main Theorem (K.)

The structure of the DDW-$\boldsymbol{m}$-degrees of real-valued functions looks like the following figure:



The DDW-$\boldsymbol{m}$-degrees form a semi-well-order of height $\boldsymbol{\Theta}$.
($\boldsymbol{\Theta}$ = the least ordinal $\alpha > 0$ s.t. there is no surjection from $\mathbb{R}$ onto α.)

For a limit ordinal $\xi < \Theta$ and finite $n < \omega$,

- the DDW-$\boldsymbol{m}$-rank $\xi + 3n + c_\xi$ consists of two incomparable degrees
- each of the other ranks consists of a single degree.

Here, $c_\xi = 2$ if $\xi = 0$; $c_\xi = 1$ if $\mathbf{cf}(\xi) = \omega$; and $c_\xi = 0$ if $\mathbf{cf}(\xi) \geq \omega_1$.

Theorem (K.-Montalbán; 201x)

The Wadge degrees ≈ the "natural" many-one degrees.

DDW defined T-reducibility for $\mathbb{R}$-valued functions as parallel continuous (strong) Weihrauch reducibility ($f \leq_T g$ iff $f \leq^c_{sW} \widehat{g}$):

$$f \leq_T g \iff (\exists H, K)(\forall x)\ K(x, (g(H(n,x)))_{n\in\mathbb{N}}) = f(x).$$

2nd Main Theorem (K.)

The DDW T-degrees ≈ the "natural" Turing degrees.

(Steel '82; Becker '88) The "natural" Turing degrees form a well-order of type Θ. Hence, the DDW T-degrees (of nonconst. functions) form a well-order of type Θ.

(The DDW T-rank of a Baire class function coincides with 2+ its Baire rank)

More Theorems... (with Westrick)

There are many other characterizations of DDW T-degrees, e.g., relative computability w.r.t. point-open topology on the space $\mathbb{R}^{(2^\omega)}$.

The 1st Main Theorem

- Pointclass: $\Gamma \subseteq \omega^\omega$
- Dual: $\check{\Gamma} = \{\omega^\omega \setminus A : A \in \Gamma\}$.
- A pointclass Γ is selfdual iff $\Gamma = \check{\Gamma}$.
- For $A, B \subseteq \omega^\omega$, A is Wadge reducible to B ($A \leq_w B$) if

$$(\exists \theta \text{ continuous})(\forall X \in \omega^\omega)\ X \in A \iff \theta(X) \in B.$$

- $A \subseteq \omega^\omega$ is selfdual if $A \equiv_w \omega^\omega \setminus A$.
- $A \subseteq \omega^\omega$ is selfdual iff $\Gamma_A = \{B \in \omega^\omega : B \leq_w A\}$ is selfdual.

$\utilde{\Delta}^i_\alpha$ is selfdual, but $\utilde{\Sigma}^i_\alpha$ and $\utilde{\Pi}^i_\alpha$ are nonselfdual.

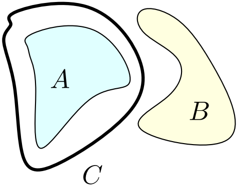
Theorem (Wadge; Martin-Monk 1970s)

The Wadge degrees are semi-well-ordered.

In particular, nonselfdual pairs are well-ordered, say $(\Gamma_\alpha, \check{\Gamma}_\alpha)_{\alpha<\Theta}$ where Θ is the height of the Wadge degrees.

A pointclass $\boldsymbol{\Gamma}$ has the separation property if

$$(\forall A, B \in \boldsymbol{\Gamma})\ [A \cap B = \emptyset \implies (\exists C \in \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}})\ A \subseteq C \ \&\ B \cap C = \emptyset]$$



Example (Lusin 1927, Novikov 1935, and others)

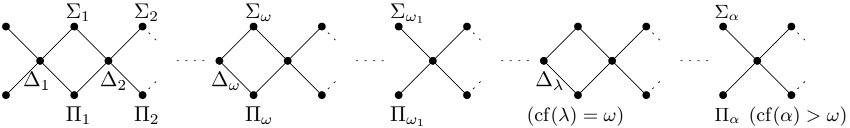
- $\boldsymbol{\Pi}^0_\alpha$ has the separation property for any $\alpha < \omega_1$.
- $\boldsymbol{\Sigma}^1_1$ and $\boldsymbol{\Pi}^1_2$ have the separation property.
- (PD) $\boldsymbol{\Sigma}^1_{2n+1}$ and $\boldsymbol{\Pi}^1_{2n+2}$ have the separation property.

Nonselfdual pairs are well-ordered, say $(\mathbf{\Gamma}_\alpha, \check{\mathbf{\Gamma}}_\alpha)_{\alpha<\Theta}$.

Theorem (Van Wasep 1978; Steel 1981)

Exactly one of $\mathbf{\Gamma}_\alpha$ and $\check{\mathbf{\Gamma}}_\alpha$ has the separation property.

- $\mathbf{\Pi}_\alpha$: the one which has the separation property
- $\mathbf{\Sigma}_\alpha$: the other one
- $\mathbf{\Delta}_\alpha = \mathbf{\Sigma}_\alpha \cap \mathbf{\Pi}_\alpha$



Example

$\mathbf{\Delta}_1$ = clopen ($\mathbf{\Delta}^0_1$); $\mathbf{\Sigma}_1$ = open ($\mathbf{\Sigma}^0_1$); $\mathbf{\Pi}_1$ = closed ($\mathbf{\Pi}^0_1$);

$\mathbf{\Delta}_\alpha$, $\mathbf{\Sigma}_\alpha$, $\mathbf{\Pi}_\alpha$ $(\alpha < \omega_1)$: the α^{th} level of the Hausdorff difference hierarchy

$\mathbf{\Sigma}_{\omega_1} = F_\sigma$ ($\mathbf{\Sigma}^0_2$); $\mathbf{\Pi}_{\omega_1} = G_\delta$ ($\mathbf{\Pi}^0_2$)

Example

- $\underset{\sim}{\mathbf{\Sigma}}^0_2$, $\underset{\sim}{\mathbf{\Pi}}^0_2$: Wadge-rank ω_1.
- $\underset{\sim}{\mathbf{\Sigma}}^0_3$, $\underset{\sim}{\mathbf{\Pi}}^0_3$: Wadge-rank $\omega_1^{\omega_1}$.
- $\underset{\sim}{\mathbf{\Sigma}}^0_n$, $\underset{\sim}{\mathbf{\Pi}}^0_n$: Wadge-rank $\omega_1 \uparrow\uparrow n$ (the n^{th} level of the superexp hierarchy)
- $\varepsilon_0[\omega_1] := \lim_{n\to\infty}(\omega_1 \uparrow\uparrow n)$: Its cofinality is ω.
  Hence, the class of rank $\varepsilon_0[\omega_1]$ is selfdual.
  Moreover, $\mathbf{\Delta}_{\varepsilon_0[\omega_1]}$ is far smaller than $\underset{\sim}{\mathbf{\Delta}}^0_\omega$.
- (Wadge) $\varepsilon_{\omega_1}[\omega_1]$: the ω_1^{th} fixed point of the exp. of base ω_1.
  $\underset{\sim}{\mathbf{\Sigma}}^0_\omega$, $\underset{\sim}{\mathbf{\Pi}}^0_\omega$: Wadge-rank $\varepsilon_{\omega_1}[\omega_1]$.

Example

- (Wadge) The Veblen hierarchy of base ω_1:
  $\phi_\alpha(\gamma)$: the γ^{th} ordinal closed under $+$, $\sup_{n\in\omega}$, and $(\phi_\beta)_{\beta<\alpha}$.
- ϕ_0 enumerates $1, \omega_1, \omega_1^2, \omega_1^3, \dots, \omega_1^{\omega+1}, \omega_1^{\omega+2}, \dots$
- ϕ_1 enumerates $1, \varepsilon_{\omega_1}[\omega_1], \dots$
- $\mathbf{\Sigma}^0_{\omega^\alpha}$, $\mathbf{\Pi}^0_{\omega^\alpha}$: Wadge-rank $\phi_\alpha(1)$ ($0 < \alpha < \omega_1$).
- $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_1$: Wadge-rank $\sup_{\xi<\omega_1} \phi_\xi(1)$.

$\mathbb{T} = \{0, 1, \perp\}$: Plotkin's order; $\perp < 0, 1$. For $A, B : \omega^\omega \to \mathbb{T}$, A is $\mathbb{T}$-m-Wadge reducible to B ($A \leq^{\mathbb{T}}_{mw} B$) if for all $n \in \mathbb{N}$ there are $m \in \mathbb{N}$ and a continuous function $\theta : \omega^\omega \to \omega^\omega$ such that

$$(\forall X \in \omega^\omega)\ A(n^\frown X) \leq B(m^\frown \theta(X)).$$

A is σ-join-reducible if $A \upharpoonright n <_w A$ for any $n \in \omega$.

For a function $f : \omega^\omega \to \mathbb{R}$, define $Lev_f : \omega^\omega \to \mathbb{T}$ as follows: For any $p, q \in \mathbb{Q}$ with $p < q$

$$Lev_f(\langle p, q\rangle^\frown X) = \begin{cases} 1 & \text{if } q \leq f(X), \\ 0 & \text{if } f(X) \leq p, \\ \perp & \text{if } p < f(X) < q. \end{cases}$$

A pair $\langle p, q\rangle$ is identified with a natural number in an effective manner.

Remark: $f \leq_m g$ iff $Lev_f \leq^{\mathbb{T}}_{mw} Lev_g$.

Main Lemma

$\forall f : \omega^\omega \to \mathbb{R}$, $\exists \xi < \Theta$ s.t. exactly one of the following holds.

- ***Lev***$_f$ is $\mathbf{\Sigma}_\xi$-complete.
- ***Lev***$_f$ is $\mathbf{\Pi}_\xi$-complete.
- ***Lev***$_f$ is $\mathbf{\Delta}_\xi$-complete and σ-join-reducible.
- ***Lev***$_f$ is $\mathbf{\Delta}_\xi$-complete and σ-join-irreducible.

Moreover, each of the above class contains exactly one $\mathbb{T}$-$\boldsymbol{m}$-W-degree.

This is NOT trivial because there are many more $\mathbb{T}$-$\boldsymbol{m}$-W-degrees:

The separation property plays a key role to show that the other degrees cannot be realized by functions of the form ***Lev***$_f$.

1st Main Theorem (K.)

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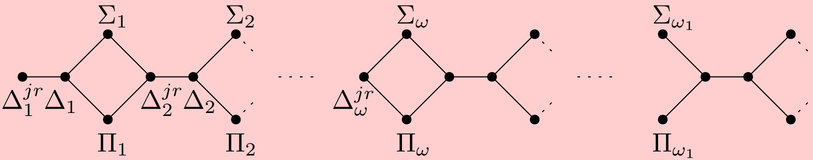

The DDW-m-degrees form a semi-well-order of height Θ.
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Here, $c_\xi = 2$ if $\xi = 0$; $c_\xi = 1$ if $\mathbf{cf}(\xi) = \omega$; and $c_\xi = 0$ if $\mathbf{cf}(\xi) \geq \omega_1$.

Fact: every Polish $\mathcal{X}$ admits an open continuous surjection $\delta : \omega^\omega \to \mathcal{X}$.

Corollary (Polish version of DDW)

$f : \mathcal{X} \to \mathbb{R}$ Baire-one; $\delta : \omega^\omega \to \mathcal{X}$ open continuous surjection

1. $\alpha(f) = \xi + 1$ and either f is left- or right-sided
   $\iff$ $\mathbf{Lev}_{f \circ \delta}$ is either $\mathbf{\Sigma}_\xi$- or $\mathbf{\Pi}_\xi$-complete.
2. $\alpha(f) = \xi$ and f is two-sided
   $\iff$ $\mathbf{Lev}_{f \circ \delta}$ is $\mathbf{\Delta}^0_\xi$-complete and σ-join-irreducible.
3. For successor ξ, $\alpha(f) = \xi$ and f is free one-sided
   $\iff$ $\mathbf{Lev}_{f \circ \delta}$ is $\mathbf{\Delta}^0_\xi$-complete and σ-join-reducible.
4. For limit ξ, $\alpha(f) = \xi$ and f is one-sided
   $\iff$ $\mathbf{Lev}_{f \circ \delta}$ is $\mathbf{\Delta}^0_\xi$-complete and σ-join-reducible.

Note that $\mathbf{\Delta}^0_\xi$, $\mathbf{\Sigma}^0_\xi$, and $\mathbf{\Pi}^0_\xi$ are the corresponding pointclasses in the ξ-th rank of the Hausdorff difference hierarchy.

The 2nd Main Theorem

- Natural Solution to Post's Problem:
  Is there a "*natural*" intermediate c.e. Turing degree?
- Natural degrees should be relativizable and degree invariant:
  - (Relativizability) It is a function $f : 2^\omega \to 2^\omega$.
  - (Degree-Invariance) $X \equiv_T Y$ implies $f(X) \equiv_T f(Y)$.
- (Sacks 1963) Is there a degree invariant c.e. operator which always gives us an intermediate Turing degree?
- (Lachlan 1975) There is no uniformly degree invariant c.e. operator which always gives us an intermediate Turing degree.
- (The Martin Conjecture; a.k.a. the **5**$^{\text{th}}$ Victoria-Delfino problem)
  - Degree invariant increasing functions are well-ordered,
  - and each successor rank is given by the Turing jump.
- (Cabal) The VD problems **1**-**5** appeared in 1978; the VD problems **6**-**14** in 1988. Only the **5**$^{\text{th}}$ and **14**$^{\text{th}}$ are still unsolved (the **14**$^{\text{th}}$ asks whether $\mathbf{AD}^+ = \mathbf{AD}$).
- (Steel 1982) The Martin Conjecture holds true for uniformly degree invariant functions.

(Hypothesis) Natural degrees are relativizable and degree-invariant.

- $f : 2^\omega \to 2^\omega$ is uniformly degree invariant (UI) if there is a function $u : \omega^2 \to \omega^2$ such that for all $X, Y \in 2^\omega$,

$$X \equiv_T Y \text{ via } (i, j) \implies f(X) \equiv_T f(Y) \text{ via } u(i, j).$$

- $f : 2^\omega \to 2^\omega$ is uniformly order preserving (UOP) if there is a function $u : \omega \to \omega$ such that for all $X, Y \in 2^\omega$,

$$X \leq_T Y \text{ via } e \implies f(X) \leq_T f(Y) \text{ via } u(e).$$

- f is Turing reducible to g on a cone ($f \leq_T^\nabla g$) if

$$(\exists C \in 2^\omega)(\forall X \geq_T C)\ f(X) \leq_T g(X) \oplus C.$$

Theorem (Steel 1982; Slaman-Steel 1988; Becker 1988)

- The $\equiv_T^\nabla$-degree of UI functions form a well-order of length Θ.
- Every UI function is $\equiv_T^\nabla$-equivalent to a UOP function.

The DDW $\boldsymbol{T}$-degrees ≈ the "natural" Turing degrees.

DDW defined $\boldsymbol{T}$-reducibility for $\mathbb{R}$-valued functions as parallel continuous (strong) Weihrauch reducibility ($f \leq_T g$ iff $f \leq^c_{sW} \widehat{g}$):

$$f \leq_T g \iff (\exists H, K)(\forall x)\ K(x, (g(H(n, x)))_{n\in\mathbb{N}}) = f(x).$$

2nd Main Theorem (K.)

The identity map gives an isomorphism between the $\equiv^\nabla_T$-degrees of UOP functions and the DDW $\boldsymbol{T}$-degrees of real-valued functions.