Homological Algebra and Reverse Mathematics (a middle report)

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R is a commutative ring with 1. M, N, L etc. are R-modules. We write $M \leq_R N$ if M is an R-submodule of N. For $S \subseteq M$, $(S)_R$ is an R-submodule generated by S.

Lemma 1 (RCA_0)

The following statements are equivalent to ACA.

(1) $\forall S \subseteq M$, $(S)_R$ *exists.*

(2) $\forall M_i \leq_R M$, $\sum M_i$ *exists.*

(3) $\forall M \leq_R N$, $\forall$ *ideal* I *of* R, IM *exists.*

(4) $\forall M \leq_R N$, $M :_R N = \{a \in R : \forall n \in N(an \in M)\}$ *exists. (Especially, the annihilator* $0 :_R N$ *of* N *exists.)*

Lemma 2 (Conidis)

Krull-Azumaya lemma is proved in ACA_0*: for any finitely generated* R*-module* M, $M = J(M)M \Rightarrow M = 0$, *where* $J(M)$ *is the intersection of all maximal ideals.*

We use ACA to prove $J(M) = \{a : \forall x \in R(1 - ax) \in R^{\times}\}$ essentially.

A chain complex $\mathcal{C}$ is a chain of R-modules

$$\to M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \to,$$

with $d_i \circ d_{i+1} = 0$.

Definition 3 (RCA_0)

For an R-hom $f : M \to N$, (L, φ) is the cokernel of f if $\varphi \circ f = 0$ and for any R-hom $\varphi' : N \to L'$, if $\varphi' \circ f = 0$ then there exists a unique R-hom $u : L \to L'$ such that $u \circ \varphi = \varphi'$.

Theorem 4

The assertion that any R-hom has the cokernel, is equivalent to ACA_0 over RCA_0.

We may think the range of d_{i+1} is $\ker(d_i)$. $H_i(\mathcal{C})$ is the cokernel of d_{i+1}. Then ACA_0 seems needed when we develop homological algebra.
But, as (abstract) simplicial complexes, we often treat only complex chains of finitely generated free $\mathbb{Z}$-modules. In this case, we can define homology groups within RCA_0 by the usual way without any additional idea. We can get some simple results in this case. For example, the following result is almost trivial.

Theorem 5

The following assertion is equivalent ACA over RCA_0: any complex chain $\mathcal{C}$ of finitely generated free $\mathbb{Z}$-modules has a sequence $\langle \beta_n : n \in \mathbb{N} \rangle$ such that each β_n is the n-th Betii number.

The proof is a simple exercise.

In this talk, we would like to discuss algebraic or categorical properties of modules for more general setting of homological algebra.
The class of all R-hom from$M$ to N, say $\mathrm{Hom}_R(M,N)$ is a Π^0_1 class. For an R-hom $f : M \to N$ and $\varphi : L \to M$, we write $f^{\#}(\varphi)$ for $f \circ \varphi$. We can think of $f^{\#}$ as a "R-hom" from $\mathrm{Hom}_R(L,M)$ to $\mathrm{Hom}_R(L,N)$.
By the same way, we can define ${}^{\#}f : \mathrm{Hom}_R(N,L)$ to $\mathrm{Hom}_R(M,L)$ by ${}^{\#}f(\varphi) = \varphi \circ f$.

Proposition 6 (RCA_0)

Assume that $0 \to N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3 \to 0$ *is exact. Then* $0 \to \mathrm{Hom}_R(M,N_1) \xrightarrow{f^{\#}} \mathrm{Hom}_R(M,N_2) \xrightarrow{g^{\#}} \mathrm{Hom}_R(M,N_3)$ *and* $0 \to \mathrm{Hom}_R(N_3,M) \xrightarrow{{}^{\#}g} \mathrm{Hom}_R(N_2,M) \xrightarrow{{}^{\#}f} \mathrm{Hom}_R(N_1,M)$ *are exact.*

Note that if $N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3$ is exact, $\mathrm{Im}\, f$ exists since $\mathrm{Im}\, f{=}\mathrm{Ker}\, g$.

Proposition 7

The five lemma is proved in RCA_0*. If we assume that the existence of necessary cokernels, the snake lemma is also proved in* RCA_0*.*

Definition 8 (RCA_0)

$(F, \langle x_i : i \in I \rangle)$ *is a free R-module if* $F = \oplus_{i \in I} R x_i$*.*

Sometimes, we only write F for a free R-module, omitting the free basis $\langle x_i : i \in I \rangle$.

Definition 9 (RCA_0)

A short exact sequence $0 \to M \xrightarrow{f} N \xrightarrow{g} L \to 0$ *is split if there exists an R-hom* $\alpha : N \to M$ *such that* $\alpha \circ f = \mathrm{Id}_M$*.*

It is also equivalent to the assertion that there exists an R-hom $\beta : L \to N$ such that $g \circ \beta = \mathrm{Id}_L$. Then $N \simeq M \oplus L$.

Definition 10 (RCA_0)

(T, φ) is a tensor product of M and N if $\varphi : M \times N \to T$ is an R-bilinear function and for any R-module T' and R-bilinear function $\varphi' : M \times N \to T'$, there exists a unique R-hom $u : T \to T'$ satisfying $u \circ \varphi = \varphi'$. We write the tensor product of M and N by $M \otimes_R N$.

Theorem 11

The following assertions are pairwise equivalent over RCA_0.

1. ACA.
2. *For any two R-modules M and N, $M \otimes_R N$ exists.*
3. *For any R-module M, $M \otimes_R M$ exists.*

Proposition 12 (RCA_0)

$R \otimes_R M \simeq M$. $M \otimes_R N \simeq N \otimes_R M$,
$(\bigoplus M_i) \otimes_R N \simeq \bigoplus (M_i \otimes_R N)$ if they exist, etc.

Definition 13 (RCA_0)

$(P, F, \iota, \varepsilon)$ is a projective R-module if F is a free R-module, $0 \to P \xrightarrow{\iota} F \xrightarrow{\varepsilon} P \to 0$ is split and $\varepsilon \circ \iota = \mathrm{Id}_P$.

As before, we only write P for a projective R-module, omitting other objects.

Proposition 14 (RCA_0)

The following are equivalent to each other.

(1) *P is projective.*

(2) *For any surjecitve R-hom $g : M \to N$ and any R-hom α, there exists R-hom $\beta : P \to M$ such that $g \circ \beta = \alpha$.*

Proposition 15 (RCA_0)

$\bigoplus P_i$ is projective iff each P_i is projective.

Using Δ^0_1-indices, we formalize an effective proof and have the following fact.

Lemma 16 ($\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$)

For any R-module M, there exist a sequence of free R-modules $\langle F_i : i \in \mathbb{N}\rangle$ and a sequence of R- homomorphisms $\langle f_i : i \in \mathbb{N}\rangle$ such that $\to F_{i+1} \xrightarrow{f_{i+1}} F_i \xrightarrow{f_i} F_{i-1} \cdots \to F_0 \xrightarrow{f_0} M \to 0$ is exact.

$\mathsf{I}\Sigma^0_2$ is used to verify required properties.
Lemma 16 guarantees the existence of projective resolution for any R-module.

Lemma 17 ($RCA_0 + I\Sigma^0_2$)

Let $f : M \to N$ be an R-hom. Assume that

$$\to P_{i+1} \xrightarrow{\partial_i} P_i \to \cdots \to P_0 \xrightarrow{\varepsilon} M \to 0,$$

$$\to Q_{i+1} \xrightarrow{\partial'_i} Q_i \cdots \to\to Q_0 \xrightarrow{\varepsilon'} N \to 0$$

are projective resolutions. Then

(1) *There is a lifting $\langle f_i : i \in \mathbb{N} \rangle$ of f.*

(2) *If $\langle f_i : i \in \mathbb{N} \rangle$ and $\langle g_i : i \in \mathbb{N} \rangle$ are liftings of f, then there exists$\langle s_i : i \in \mathbb{N} \rangle$ such that $s_{i-1} \circ \partial_i + \partial'_{i+1} \circ s_i = f_i - g_i$.*

Remark. If each P_i and Q_i are finitely generated (with generators uniformly), the above is proved in RCA_0.

Let $\to P_i \xrightarrow{\partial_i} \cdots \to P_0 \xrightarrow{\epsilon} M \to 0$ be a projective resolution. Then we have

$$0 \to \mathrm{Hom}_R(P_0, N) \xrightarrow{\#\partial_1} \mathrm{Hom}_R(P_1, N) \to \mathrm{Hom}_R(P_2, N) \to \cdots .$$

Let Z_i be a Π^0_1-class of the elements α of $\mathrm{Hom}_R(P_i, N)$ such that $\alpha \circ \partial_{i+1} = 0$. Define $\alpha =_i \beta$ by $\alpha - \beta = \gamma \circ \partial_i$ for some $\gamma \in \mathrm{Hom}_R(P_i, N)$.
We define $\mathrm{Ext}^i_R(M, N)$ by $(Z_i, =_i)$ within $\mathsf{RCA}_0{+}\mathsf{I}\Sigma^0_2$. Lemma 16 tells us that we may assume that this definition is independent of projective resolutions. It is not difficult to show that $\mathrm{Ext}^i_R(M, N)$ acts a functor.
If a chain complex $\to P_i \otimes N \to \cdots \to P_0 \otimes N \to 0$ is given, we can also define a functor $\mathrm{Tor}^i_R(M, N)$ within $\mathsf{RCA}_0{+}\mathsf{I}\Sigma^0_2$.

Definition 18 (RCA_0)

An R-module I is injective if for any R-monomorphism $f : M \to N$ and any R-hom $\alpha : M \to I$, there exists an R-hom $\beta : N \to I$ such that $\beta \circ f = \alpha$.

Proposition 19 (RCA_0)

The following are equivalent to each other.

(1) ACA.

(2) *Baer's test: "For any ideal J of R and any R-hom $\alpha : J \to I$, there exists an R-hom $\beta : J \to I$ such that $\beta | J = \alpha$" implies that I is injective.*

Proposition 20 (RCA_0)

Any $\mathbb{Z}$-module M has an injective $\mathbb{Z}$-module N such that $M \leq_{\mathbb{Z}} N$.

Proposition 21 (ACA_0; Wu & Wu, 16)

If R is a P.I.D., then any submodule of a free R-module is free.

Let $\mathcal{I} = (I, \langle J(i,i') : i, i' \in I \rangle, \circ)$ be a category.
I is the set of objects and $J(i,i')$ is the set of arrows from i to i'.

A pair of sequences $(\langle M_i : i \in I \rangle, \langle \langle f_\varphi : \varphi \in J(i,i') \rangle : i, i' \in I \rangle)$ is said to be a diagram of type $\mathcal{I}$ if (1) each f_φ is an R-hom from $M_{\mathrm{dom}\varphi}$ to $M_{\mathrm{codom}\varphi}$, (2) $f_{\mathrm{Id}_i} = \mathrm{Id}_{M_i}$ for all $i \in I$, and (3) $f_{\varphi_1 \circ \varphi_2} = f_{\varphi_1} \circ f_{\varphi_2}$ for any adequate arrows φ_1, φ_2.

Then we can define the inductive limit $(L, \langle \iota_i : i \in I \rangle)$ by the usual way.

It is unique up to isomorphism if it exists. We write $\lim_{\rightarrow i \in I} M_i$ for the inductive limit. We also define the projective limit as the dual of inductive limit and write $\lim_{\leftarrow i \in I} M_i$ for it.

Proposition 22 (RCA_0)

ACA *is equivalent to the statement that any diagram has the inductive limit. Especially, the existence of the co-equalizer (or pushout) for two R-homomorphisms implies* ACA.

Proposition 23 (RCA_0)

Any diagram of a finite type has the projective limit. Especially, the equalizer (or pullack) for any two R-homomorphisms exists.

Proposition 24 (RCA_0)

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be categories. Assume that $(\langle M_{i_1 i_2} : (i_1, i_2) \in I_1 \times I_2 \rangle, \cdots)$ is a diagram of $\mathcal{I}_1 \times \mathcal{I}_2$. Then

$$\lim_{\to i_1 \in I_1} \lim_{\to i_2 \in I_2} M_{i_1 i_2} \simeq \lim_{\to i_2 \in I_2} \lim_{\to i_1 \in I_1} M_{i_1 i_2} \simeq \lim_{\to (i_1, i_2) \in I_1 \times I_2} M_{i_1 i_2},$$

if the above limits exist.

A category $\mathcal{I} = (I, \langle J(i, i') : i, i' \in I \rangle, \circ)$ is filtered if (1) I is not empty, (2) $\forall i, i' \exists i'' (J(i, i'') \neq \emptyset \wedge J(i', i'') \neq \emptyset)$, and (3) $\forall \varphi_1, \varphi_2 \in J(i, i') \exists \mu \in J(i', j)$ such that $\mu \circ \varphi_1 = \mu \circ \varphi_2$.

Proposition 25 (RCA_0)

Let $(\langle M_i : i \in I \rangle, \cdots)$ be a diagram of a filtered category $\mathcal{I}$. Then

$$\lim_{\to i \in I} M_i \simeq \coprod M_i / \sim,$$

where $x \sim y \Leftrightarrow \varphi(x) = \psi(y)$ for some $\varphi \in J(i, i^*)$ and $\psi \in J(i', i^*)$, for $x \in M_i$ and $y \in M_{i'}$.

Note that the above proposition doesn't mention the existence of $\lim_{\rightarrow i \in I} M_i$ and $\coprod M_i / \sim$.

Proposition 26 (RCA_0)

Let $\mathcal{I}_1$ be a filtered category and $\mathcal{I}_2$ be a finite category. Assume that $(\langle M_{i_1 i_2} : (i_1, i_2) \in I_1 \times I_2 \rangle, \cdots)$ is a diagram of $\mathcal{I}_1 \times \mathcal{I}_2^{op}$. Then

$$\lim_{\rightarrow i_1 \in I_1} \lim_{\leftarrow i_2 \in I_2} M_{i_1 i_2} \simeq \lim_{\leftarrow i_2 \in I_2} \lim_{\rightarrow i_1 \in I_1} M_{i_1 i_2},$$

if the above limits exist.

Proposition 27 (ACA_0)

Let $\mathcal{M}^k = (\langle M_i^k : i \in I \rangle, \cdots)$ is a diagram of $\mathcal{I}$ for $k = 1, 2, 3$. Let $\langle g_i \rangle : \mathcal{M}^1 \rightarrow \mathcal{M}^2$ and $\langle h_i \rangle : \mathcal{M}^2 \rightarrow \mathcal{M}^3$ be homomorphisms. Assume that each $M_i^1 \rightarrow M_i^2 \rightarrow M_i^3$ is exact. Then $\lim_{\rightarrow i \in I} M_i^1 \rightarrow \lim_{\rightarrow i \in I} M_i^2 \rightarrow \lim_{\rightarrow i \in I} M_i^3$.

Proposition 28 (RCA_0)

Let $\mathcal{I}_1$ be a filtered category and $\mathcal{I}_2$ be a finite category. Assume that $(\langle M_{i_1 i_2} : (i_1, i_2) \in I_1 \times I_2 \rangle, \cdots)$ is a diagram of $\mathcal{I}_1 \times \mathcal{I}_2^{op}$. Then

$$\lim_{\rightarrow i_1 \in I_1} \lim_{\leftarrow i_2 \in I_2} M_{i_1 i_2} \simeq \lim_{\leftarrow i_2 \in I_2} \lim_{\rightarrow i_1 \in I_1} M_{i_1 i_2},$$

if the above limits exist.

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