Cantor's uniqueness theorems and countable closed sets

Wim Veldman
Radboud University Nijmegen

Second Workshop on Mathematical Logic and its Applications
Kanazawa, 5-9 March 2018

B. Riemann

For which functions $F : [-\pi, \pi] \to \mathcal{R}$ do there exists reals $b_0, a_1, b_1, \ldots$ such that, for all x in $[-\pi, \pi]$,

$$F(x) = \frac{b_0}{2} + \sum_{n>0} a_n \sin nx + b_n \cos nx?$$

(*Habilitationsschrift*, 1854, 1867).

Riemann's starting point:

Let $b_0, a_1, b_1, \ldots$ satisfy:

$$\text{for each } x,\ \lim_{n\to\infty}(a_n \sin nx + b_n \cos nx) = 0.$$

Now define:

$$F(x) := \frac{b_0}{2} + \sum_{n>0} a_n \sin nx + b_n \cos nx,$$

$$G(x) := \frac{1}{4} b_0 x^2 + \sum_{n>0} \frac{-a_n}{n^2} \sin nx + \frac{-b_n}{n^2} \cos nx$$

F mostly is only a *partial* function, but G is defined everywhere.

Symmetric derivatives

$$D^2 H(x) = \lim_{h \to 0} \frac{H(x+h) + H(x-h) - 2H(x)}{h^2}$$

$$D^1 H(x) = \lim_{h \to 0} \frac{H(x+h) + H(x-h) - 2H(x)}{h}$$

Riemann proved, **constructively**, for each x in $(-\pi, \pi)$,

1. if $F(x)$ is defined, then $F(x) = D^2 G(x)$, and,
2. in any case, $D^1 G(x) = 0$.

Cantor: Uniqueness?

For every function $F : [-\pi, \pi] \to \mathcal{R}$ there exists **at most one** infinite sequence of reals $b_0, a_1, b_1, \ldots$ such that, for every x in $[-\pi, \pi]$,

$$F(x) = \frac{b_0}{2} + \sum_{n>0} a_n \sin nx + b_n \cos nx.$$

that is:

for every infinite sequence of reals $b_0, a_0, b_1, \ldots$,

if, for every x in $[-\pi, \pi]$, $\frac{b_0}{2} + \sum_{n>0} a_n \sin nx + b_n \cos nx = 0$,

then $b_0 = 0$ and $\forall n > 0[a_n = b_n = 0]$.

The Cantor-Schwarz Lemma

Assume $G : [a, b] \to \mathcal{R}$ is continuous, $G(a) = G(b) = 0$ and:
$\forall x \in (a, b)[D^2G(x)\downarrow]$.
Then:

(i) For all $\varepsilon > 0$, if $\exists x \in [a, b][G(x) = \varepsilon]$, then $\exists z \in (a, b)[D^2G(z) \leq -\frac{2\varepsilon}{(b-a)^2}]$, and,

(ii) if $\forall x \in (a, b)[D^2G(x) = 0]$, then $\forall x \in [a, b][G(x) = 0]$.

The **intuitionistic** proof uses the **Fan Theorem**.

Brouwer's Fan Theorem **FT**

$B \subseteq \{0,1\}^*$ is a bar in $\mathcal{C} = \{0,1\}^{\mathbb{N}} := \forall \alpha \in \mathcal{C} \exists n [\overline{\alpha} n \in B]$.

If B is a bar in $\mathcal{C}$, then some finite $B' \subseteq B$ is a bar in $\mathcal{C}$.

FT implies:

1. Als $[0,1] \subseteq \bigcup_{n \in \mathbb{N}} (a_n, b_n)$, dan $\exists N [[0,1] \subseteq \bigcup_{n < N} (a_n, b_n)]$.
2. If $f : [0,1] \to \mathcal{R}$ is continuous at every point, then f is continuous uniformly on $[0,1]$ and $Ran(f)$ has a least upper bound.

The proof by Schwarz

Assume: $x \in [a,b]$ and $G(x) = \varepsilon > 0$.

Define $H : [a,b] \to \mathcal{R}$:

$$H(y) = G(y) - \varepsilon \frac{(b-y)(y-a)}{(b-a)^2}.$$

Note: $H(x) \geq \frac{3}{4}\varepsilon$.

And: $\forall y \in (a,b)[D^2H(y)\downarrow \wedge D^2H(y) = D^2G(y) + \frac{2\varepsilon}{(b-a)^2}]$.

Weierstrass: find $\mathbf{y_0}$ such that $\forall y \in [a,b][H(y) \leq H(\mathbf{y_0})]$

and: $D^2H(\mathbf{y_0}) \leq 0$, and: $D^2G(\mathbf{y_0}) \leq -\frac{2\varepsilon}{(b-a)^2}$.

An intuitionistic way out

For every ρ, define $H_\rho : [a, b] \to \mathcal{R}$:

$$H_\rho(y) = H(y) + \rho \frac{y - a}{b - a}.$$

Note: for every ρ, for all y in (a, b):

$D^2 H_\rho(y) \downarrow \wedge D^2 H_\rho(y) = D^2 H(y) = D^2 G(y) + \frac{2\varepsilon}{(b-a)^2}$.

Construct ρ and $\mathbf{y_0}$ in $[a, b]$ such that

$\forall y \in [a, b][H_\rho(y) \leq H_\rho(\mathbf{y_0})]$.

The construction uses:

if $f : [c, d] \to \mathcal{R}$ is continuous then $\sup_{[c,d]} \downarrow$.

Cantor 1870: Uniqueness proven!

$$F(x) := \frac{b_0}{2} + \sum_{n>0} a_n \sin nx + b_n \cos nx,$$

$$G(x) := \frac{1}{4} b_0 x^2 + \sum_{n>0} \frac{-a_n}{n^2} \sin nx + \frac{-b_n}{n^2} \cos nx$$

Assume: $\forall x \in [-\pi, \pi][F(x) = 0]$.

Riemann: $\forall x \in [-\pi, \pi][D^2 G(x) = 0]$.

Cantor-Schwarz: G is linear.

Provisional assumption: The sequence $b_0, a_1, b_1, \ldots$ is **bounded**.

Then: **uniform convergence** of G and: $b_0 = 0$ and $\forall n > 0[a_n = b_n = 0]$.

Kronecker: no need to assume boundedness

Do not assume: The sequence $b_0, a_1, b_1, \ldots$ is bounded.

Let $x \in [-\pi, \pi]$ be given. Define, for every t in $[-\pi, \pi]$,

$$K(t) = F(x+t) + F(x-t)$$

$$K(t) := b_0 + 2\sum_{n>0}(a_n \sin nx + b_n \cos nx)\cos nt$$

$(\sin(nx+nt) + \sin(nx-nt) = 2\sin nx \cos nt,$
$\cos(nx+nt) + \cos(nx-nt) = 2\cos nx \cos nt)$.

For every t in $[-\pi, \pi]$, $K(t) = 0$, so: $b_0 = 0$ and, for every $n > 0$, $a_n \sin nx + b_n \cos nx = 0$. For all x in $[-\pi, \pi]$!

Conclude: $b_0 = 0$ and $\forall n > 0[a_n = b_n = 0]$.

Cantor's different way

Cantor proved: Riemann's starting point:

$$\text{for all } x,\ \lim_{n\to\infty}(a_n \sin nx + b_n \cos nx) = 0.$$

implies:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$$

and:

The sequence $b_0, a_1, b_1, \ldots$ is bounded.

Cantor's proof is **(irreparably?) non-constructive**.
However, one may obtain the desired conclusion from **Brouwer's Continuity Principle**.

Cantor goes on

$\mathcal{X} \subseteq [-\pi, \pi]$ **guarantees uniqueness** $:=$
for every infinite sequence $b_0, a_1, b_1, \ldots$ of reals,
if $\forall x \in \mathcal{X}[F(x) := \frac{b_0}{2} + \sum_{n>0} a_n \sin nx + b_n \cos nx = 0]$,
then $b_0 = 0$ and, $\forall n > 0[a_n = b_n = 0]$.

Cantor 1871: Every co-finite $\mathcal{X} \subseteq [-\pi, \pi]$ guarantees uniqueness.

For: If $G : [a, b] \to \mathcal{R}$ and $a < c < b$ en G is linear on $[a, c]$ and on $[c, b]$ and $D^1 G(c) = 0$, then G is linear on $[a, b]$.

The co-derivative

$H : (a, b) \to \mathcal{R}$ is **locally linear on $\mathcal{G} \subseteq (a, b)$**$:=$
$\forall x \in \mathcal{G} \exists n [H$ is linear on $(x - \frac{1}{2n}, x + \frac{1}{2^n}) \subseteq \mathcal{G}]$.

If H is locally linear on (a, b), then H is linear on (a, b).

$\mathcal{G} \subseteq \mathcal{R}$ is **open** $:= \forall x \in \mathcal{G} \exists n [(x - \frac{1}{2^n}, x + \frac{1}{2^n}) \subseteq \mathcal{G}]$.
$x \in \mathcal{G}^+ :=$ there exist n, y such that: $|x - y| < \frac{1}{2^n}$ and

$$\forall z [(|x - z| < \frac{1}{2^n} \ \wedge \ z \#_{\mathcal{R}} \ y) \to z \in \mathcal{G}] :$$

all points in some neighbourhood of x are in $\mathcal{G}$ **with one clearly indicated possible exception**.

$\mathcal{G}^+$: the **(first) co-derivative (extension) of** $\mathcal{G}$.

Cantor's big step

Assume: $G : [-\pi, \pi] \to \mathcal{R}$ en $\forall x \in [-\pi, \pi][D^1 G(x) = 0]$.

Let $\mathcal{G} \subseteq (-\pi, \pi)$ be open.

If G is locally linear on $\mathcal{G}$, then G is locally linear on $\mathcal{G}^+$.

Define: $\mathcal{G}^{(0)} = \mathcal{G}$ and, for each n, $\mathcal{G}^{(n+1)} = (\mathcal{G}^{(n)})^+$.

If G is locally linear on $\mathcal{G}$, then G is locally linear on each $\mathcal{G}^{(n)}$.

$[-\pi, \pi]$ is **swiftly full of** $\mathcal{G} := \exists n[\mathcal{G}^{(n)} = (-\pi, \pi)]$.

If $[-\pi, \pi]$ is swiftly full of $\mathcal{G}$, $\mathcal{G}$ guarantees uniqueness.

Transfinite extension

Let $\mathcal{G} \subseteq [-\pi, \pi]$ be open.

We define the collection $Ext_{\mathcal{G}}$ of the co-derivative extensions of $\mathcal{G}$ **inductively**:

1. $\mathcal{G} \in Ext_{\mathcal{G}}$.
2. For all $\mathcal{H}$ in $Ext_{\mathcal{G}}$, also $\mathcal{H}^+ \in Ext_{\mathcal{G}}$.
3. for every infinite sequence $\mathcal{H}_o, \mathcal{H}_1, \ldots$ of elements of $Ext_{\mathcal{G}}$, also $\bigcup_{n \in \mathbb{N}} \mathcal{H}_n \in Ext_{\mathcal{G}}$.
4. Nothing more.

If $\forall x \in [-\pi, \pi][D^1 G(x) = 0]$ and G is locally linear on $\mathcal{G}$, then G is locally linear on every $\mathcal{H}$ in $Ext_{\mathcal{G}}$.

We define: $[-\pi, \pi]$ is **eventually full of** $\mathcal{G}$:= $(-\pi, \pi) \in Ext_{\mathcal{G}}$.

If $[-\pi, \pi]$ is eventually full of $\mathcal{G}$, then $\mathcal{G}$ guarantees uniqueness.

Countable and almost-countable

Assume: $\mathcal{X} \subseteq \mathcal{R}$ and $f : \mathbb{N} \to \mathcal{R}$.

f **enumerates** $\mathcal{X} := \forall x \in \mathcal{X} \exists n[x = f(n)]$.

f **almost-enumerates** $\mathcal{X} :=$
$\forall x \in \mathcal{X} \forall \gamma \in \mathcal{N} \exists n[\exists n[|f(n) - x| < \frac{1}{2^{\gamma(n)}}]$.

If $\mathcal{G}$ is eventually full in $[-\pi, \pi]$, then $[-\pi, \pi] \setminus \mathcal{G}$ is almost-enumerable.

this follows from:

For every $\mathcal{X}$ in $Ext_{\mathcal{G}}$, for all a, b such that $-\pi \leq a < b \leq \pi$, if $[a, b] \subseteq \mathcal{X}$, then $[a, b] \setminus \mathcal{G}$ is almost-enumerable.

Locatedness

$\mathcal{X} \subseteq [-\pi, \pi]$ is **located**:=
for every x in $[-\pi, \pi]$ one may determine
$d(x, \mathcal{X})$, *the distance from x to $\mathcal{X}$*:

1. $\forall y \in \mathcal{X}[d(x, \mathcal{X}) \leq |y - x|]$.
2. $\forall \varepsilon > 0 \exists y \in \mathcal{X}[|y - x| < d(x, \mathcal{X}) + \varepsilon]$.

Assume: $\mathcal{G} \subseteq \mathcal{R}$ is open.
If $\mathcal{F} := [-\pi, \pi] \setminus \mathcal{G}$ is located, then $\mathcal{F}$ may be covered by a fan and every open cover of $\mathcal{F}$ has as a finite subcover.

Bar induction

Assume: $B, C \subseteq \mathbb{N}^*$.

B is a **bar in $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$** $:= \forall\alpha \in \mathcal{N} \exists n[\overline{\alpha}n \in B]$.

B is **monotone** $:= \forall c[c \in B \to \forall m[c * (m) \in B]]$.

C is **inductive**$:= \forall c[\forall m[c * (m) \in C] \to c \in C]$.

BI$_M$:

If B is monotone and a bar in $\mathcal{N}$, and $B \subseteq C$, and C is inductive, then $(\ ) \in C$.

A reversal

Assume: $\mathcal{G} \subseteq (-\pi, \pi)$ *is open.*

If $[-\pi, \pi] \setminus \mathcal{G}$ *is located and almost-enumerable, then* $\mathcal{G}$ *is eventually full in* $[-\pi, \pi]$*:* $(-\pi, \pi) \in Ext_{\mathcal{G}}$.

Why?

Let f be an almost-enumeration of $\mathcal{F} := [-\pi, \pi] \setminus \mathcal{G}$.

$$\forall x \in \mathcal{F} \forall \gamma \exists n [|f(n) - x| \leq \frac{1}{2^{\gamma(n)}}]$$

$$\forall \gamma \forall x \in \mathcal{F} \exists n [|f(n) - x| \leq \frac{1}{2^{\gamma(n)}}]$$

$$\forall \gamma \in \mathcal{N} \exists N \forall x \in \mathcal{F} \exists n \leq N [|f(n) - x| \leq \frac{1}{2^{\gamma(n)}}].$$

$c \in B := \forall x \in \mathcal{F} \exists n < length(c)[|f(n) - x| \leq \frac{1}{2^{c(n)}}]$.

B is a monotone bar in $\mathcal{N}$.

$\mathcal{H}_c := \{x \in (-\pi, \pi) | \forall n < length(c)[|f(n) - x| > \frac{1}{2^{c(n)}}]\}$.

$c \in C := \exists \mathcal{X} \in Ext_{\mathcal{G}}[\mathcal{H}_c \subseteq \mathcal{X}]$.

If $c \in B$, then $\mathcal{H}_c = \emptyset$. Conclude: $B \subseteq C$.

Assume: $c \in \mathbb{N}^*$ and: $\forall m[c * (m) \in C]$.

Find $n := length(c)$.

Find $\mathcal{X}_0, \mathcal{X}_1, \ldots$ in $Ext_{\mathcal{G}}$ such that, for each m,

$\mathcal{H}_{c*(m)} = \mathcal{H}_c \cap \{x \in (-\pi, \pi) || x - f(n)| > \frac{1}{2^m}\} \subseteq \mathcal{X}_m$.

Observe: $\mathcal{H}_c \subseteq (\bigcup_{p \in \mathbb{N}} \mathcal{X}_p)^+$.

Conclude: C is inductive.

Conclude: $(\ ) \in C$ and: $(-\pi, \pi) \in Ext_{\mathcal{G}}$.

Cantor-Schwarz-Bernstein-Young (1908/1909)

Assume: $\mathcal{X} \subseteq (a, b)$ is **co-enumerable**:

$$\exists f : \mathbb{N} \to (a, b) \forall x \in (a, b)[\forall n[f(n) \# x] \to x \in \mathcal{X}].$$

Let $G : [a, b] \to \mathcal{R}$ be continuous and $G(a) = G(b) = 0$ and $\forall x \in \mathcal{X}[D^2 G(x) \downarrow]$ and: $\forall x \in (a, b)[D^1 G(x) = 0]$.

Then:

(i) For every $\varepsilon > 0$, if $\exists x \in [a, b][G(x) = \varepsilon]$, then $\exists z \in \mathcal{X}[D^2 G(z) \leq -\frac{2\varepsilon}{(b-a)^2}]$, and,

(ii) if $\forall x \in \mathcal{X}[D^2 G(x) = 0]$, then $\forall x \in [a, b][G(x) = 0]$.

The proof

Assume: $x \in [a, b]$ and $G(x) = \varepsilon > 0$.
Define $H : [a, b] \to \mathcal{R}$:

$$H(y) = G(y) - \varepsilon \frac{(b - y)(y - a)}{(b - a)^2}.$$

For each ρ, define: $H_\rho : [a, b] \to \mathcal{R}$:

$$H_\rho(y) = H(y) + \rho \frac{y - a}{b - a}.$$

Assume: $\rho_0 \# \rho_1$ and $\forall x \in [a, b][H_{\rho_0}(x) \leq H_{\rho_0}(y_0)]$ and $\forall x \in [a, b][H_{\rho_1}(x) \leq H_{\rho_1}(y_1)]$.

Note: $H'_{\rho_0}(y_0) = 0$ en $H'_{\rho_1}(y_1) = 0$,

and: $H'(y_0) = -\frac{\rho_0}{b-a} \# -\frac{\rho_1}{b-a} = H'(y_1)$,

Conclude: $y_0 \# y_1$.

Cantor defeated

Bernstein/Young, *intuitionistically*:

> *Every co-enumerable* $\mathcal{X} \subseteq [-\pi, \pi]$ *guarantees uniqueness.*

classically:

> *Every enumerable* $\mathcal{X} \subseteq [-\pi, \pi]$ *is a set of uniqueness.*