# On a computational interpretation of sequent calculus for modal logic S4

## Yosuke Fukuda

Graduate School of Informatics, Kyoto University

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  -modality plays an essential role:
  - □*A* means the set of programs which "encode" programs of type *A*

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To do this, a sequent calculus and its corresponding calculus for intuitionistic S4 are proposed

proof-theoretically based on:

 a modal sequent calculus and the G3-style system [Troelstra&Schwichtenberg '96]

a higher-arity modal natural deduction [Pfenning&Davies '01]

type-theoretically based on:

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 the Curry–Howard correspondence for a G3-style sequent calc. [Ohori '99]

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**1** proof-theoretically based on:

 a modal sequent calculus and the G3-style system [Troelstra&Schwichtenberg '96]

- a higher-arity modal natural deduction [Pfenning&Davies '01]
- 2 type-theoretically based on:
  - the higher-arity modal  $\lambda$ -calculus [Pfenning&Davies '01]
  - the Curry–Howard correspondence for a G3-style sequent calc. [Ohori '99]

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# Higher-arity Sequent Calculus for intuitionistic S4

We propose a "higher-arity" sequent calc. for  $(\land, \lor, \supset, \Box)$ -fragment of intuitionistic S4, **HLJ<sub>S4</sub>**, based on [Troelstra&Schwichtenberg '96]

Definition (Formula)

$$A, B ::= p \mid A \land B \mid A \lor B \mid A \supset B \mid \Box A$$

Definition (Higher-arity judgment [Pfenning+ '01])

A judgment is defined by the following higher-arity form:

 $\Delta; \Gamma \vdash A$ 

which intuitively means  $(\bigwedge \Box \Delta) \land (\bigwedge \Gamma) \supset A$ 

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# Inference rules of HLJ<sub>S4</sub>

$$\frac{\overline{\emptyset; A \vdash A} \quad Ax}{\overline{\emptyset; A \vdash A} \quad \Delta; \Gamma \vdash B} \quad AR \quad -\frac{}{A}$$

$$\frac{\Delta; \Gamma \vdash A \land B}{\Delta; \Gamma \vdash A_1 \lor A_2} \lor R \quad \frac{\Delta; \Gamma}{\Delta; \Gamma \vdash A_1 \lor A_2} \lor R \quad \frac{\Delta; \Gamma}{\Delta; \Gamma \vdash A \supset B} \supset R \quad \frac{\Delta; \Gamma}{\Delta; \Gamma \vdash A \supset B} \supset R \quad \frac{\Delta; \Gamma}{\Delta; 0 \vdash \Box A} \quad \frac{\Box R}{\Delta; 0 \vdash \Box A} \quad \frac{\Box R}{\Delta; \Gamma, A \vdash B} \quad W \quad \frac{\Delta; \Gamma, A \vdash B}{\Delta; \Gamma, A \vdash B} \quad C \quad \frac{\Delta; \Gamma \vdash A \land B}{\Delta; \Gamma \vdash B} \quad Cut \quad -\frac{\Delta}{\Delta; \Gamma$$

$$\frac{\overline{A; \emptyset \vdash A} \Box Ax}{A; \Gamma \vdash B} \land L$$

$$\frac{\Delta; \Gamma, A_{1} \vdash B}{\Delta; \Gamma, A_{1} \land A_{2} \vdash B} \land L$$

$$\frac{\Box; \Gamma, A \vdash C}{\Delta; \Gamma, A \lor B \vdash C} \lor L$$

$$\frac{\Box; \Gamma \vdash A}{\Delta; \Gamma, A \lor B \vdash C} \supset L$$

$$\frac{\Delta, A; \Gamma \vdash B}{\Delta; \Gamma, \Box A \vdash B} \Box L$$

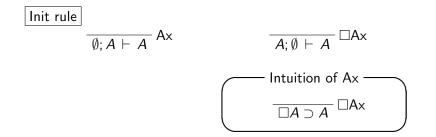
$$\frac{\Delta, A; \Gamma \vdash B}{\Delta, A; \Gamma \vdash B} \Box W$$

$$\frac{\Delta, A, A; \Gamma \vdash B}{\Delta, A; \Gamma \vdash B} \Box C$$

$$\frac{\Delta; \emptyset \vdash A}{\Delta; \Gamma \vdash B} \Box C$$

$$\frac{\Delta; \emptyset \vdash A}{\Delta; \Gamma \vdash B} \Box C$$

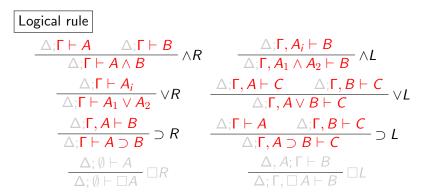




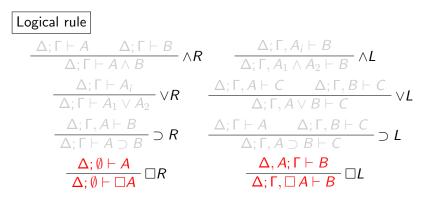
$$\begin{array}{c|c} \hline \text{Logical rule} \\ \hline \underline{\Delta; \Gamma \vdash A} & \underline{\Delta; \Gamma \vdash B} \\ \hline \Delta; \Gamma \vdash A \land B} \land R & \underline{\Delta; \Gamma, A_i \vdash B} \\ \hline \underline{\Delta; \Gamma \vdash A_i} \\ \hline \underline{\Delta; \Gamma \vdash A_i} \lor A_2 \lor R & \underline{\Delta; \Gamma, A \vdash C} & \underline{\Delta; \Gamma, B \vdash C} \\ \hline \underline{\Delta; \Gamma \vdash A_1 \lor A_2} \lor R & \underline{\Delta; \Gamma, A \lor B \vdash C} \lor L \\ \hline \underline{\Delta; \Gamma, A \vdash B} \\ \hline \underline{\Delta; \Gamma \vdash A \supset B} \supset R & \underline{\Delta; \Gamma \vdash A} & \underline{\Delta; \Gamma, B \vdash C} \\ \hline \underline{\Delta; \Gamma, A \supset B} \vdash C \\ \hline \underline{\Delta; \emptyset \vdash A} \Box R & \underline{\Delta; \Gamma, B \vdash B} \Box L \end{array}$$

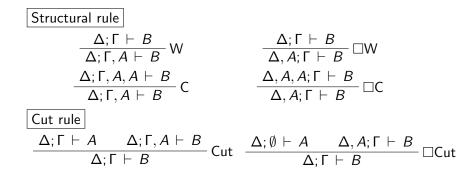
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While we can prove the cut-elimination theorem for  $HLJ_{S4}$ , the proof by the mix-elimination is problematic; because ...

$$\frac{\prod'}{\Delta; \Gamma \vdash A} \xrightarrow{\Delta'; \Gamma', A, A \vdash B}{\Delta; \Gamma', A \vdash B} \stackrel{\mathsf{C}}{\operatorname{Cut}} \Longrightarrow \xrightarrow{\Delta; \Gamma \vdash A} \xrightarrow{\Delta'; \Gamma', A, A \vdash B}{\Delta; \Delta; \Gamma \vdash B} \operatorname{Mix}$$

In the elimination procedure,

- it is "okay" if we consider the provability of the judgment; but
- it is "not okay" if we consider the construction of the judgment

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While we can prove the cut-elimination theorem for  $HLJ_{S4}$ , the proof by the mix-elimination is problematic; because ...

$$\frac{\prod'_{(\Delta;\Gamma\vdash A)}}{\Delta;\Gamma\vdash A} \xrightarrow{\Delta';\Gamma',A,A\vdash B}_{\Delta;\Gamma\vdash B} C \Longrightarrow \xrightarrow{\prod_{(\Delta;\Gamma\vdash A)}}_{\Delta;\Gamma\vdash A} \xrightarrow{\Delta';\Gamma',A,A\vdash B}_{\Delta;\Delta';\Gamma,\Gamma\vdash B} Mix$$

In the elimination procedure,

- it is "okay" if we consider the provability of the judgment; but
- it is "not okay" if we consider the construction of the judgment

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The *G3-style* [Kleene '52] [Dragalin '88] is a style of formalization to make a cut-free, or precisely, "structural-rule-free" system

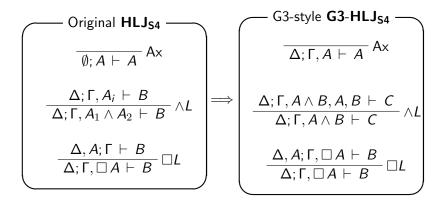
The G3-style inference rules are defined in a somewhat tricky way to derive the "height-preserving admissible" structural rules

The G3-style inference rules are defined as follows:  $\Delta$ . A:  $\Gamma \vdash A$   $\Box$ Ax  $\Delta: \Gamma A \vdash A$  Ax  $\frac{\Delta; \Gamma, A \vdash B}{\Delta; \Gamma \vdash A \supset B} \supset R \xrightarrow{\Delta; \Gamma, A \supset B \vdash A} \Delta; \Gamma, A \supset B, B \vdash C}{\Delta; \Gamma, A \supset B \vdash C} \supset L$  $\frac{-\Delta; \Gamma \vdash A}{\Delta; \Gamma \vdash A \land B} \land R \quad \frac{-\Delta; \Gamma, A \land B, A, B \vdash C}{\wedge \cdot \Gamma \land A \land B \vdash C} \land L$  $\frac{\Delta; \Gamma \vdash A_i}{\Delta; \Gamma \vdash A_1 \lor A_2} \lor R$  $\frac{\Delta; \Gamma, A \lor B, A \vdash C}{\Delta; \Gamma, A \lor B, B \vdash C} \lor L$  $\frac{\Delta, A; \Gamma, \Box A \vdash B}{\Delta; \Gamma, \Box A \vdash B} \Box L$  $\frac{\Delta; \emptyset \vdash A}{\Delta; \Gamma \vdash \Box A} \Box R$ 

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Idea: all we have to do is to get "height-preserving" structural rules



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## Lemma (Height-preserving weakening/contraction)

The followings are height-preserving admissible rules in G3-HLJ<sub>S4</sub>:

$$\frac{\Delta; \Gamma \vdash B}{\Delta; \Gamma, A \vdash B} W \qquad \qquad \frac{\Delta; \Gamma \vdash B}{\Delta, A; \Gamma \vdash B} \square W \\ \frac{\Delta; \Gamma, A, A \vdash B}{\Delta; \Gamma, A \vdash B} C \qquad \qquad \frac{\Delta, A, A; \Gamma \vdash B}{\Delta, A; \Gamma \vdash B} \square C$$

#### Theorem (Equivalence)

The provability of HLJ<sub>S4</sub> and G3-HLJ<sub>S4</sub> + Cut is equivalent

#### Theorem (Cut-elimination)

The cut rules Cut and Cut are admissible in G3-HLJ<sub>S4</sub>

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## Lemma (Height-preserving weakening/contraction)

The followings are height-preserving admissible rules in G3-HLJ<sub>S4</sub>:

$$\frac{\Delta; \Gamma \vdash B}{\Delta; \Gamma, A \vdash B} W \qquad \qquad \frac{\Delta; \Gamma \vdash B}{\Delta, A; \Gamma \vdash B} \Box W$$

$$\frac{\Delta; \Gamma, A, A \vdash B}{\Delta; \Gamma, A \vdash B} C \qquad \qquad \frac{\Delta, A, A; \Gamma \vdash B}{\Delta, A; \Gamma \vdash B} \Box C$$

### Theorem (Equivalence)

The provability of  $HLJ_{S4}$  and  $G3-HLJ_{S4} + Cut$  is equivalent

### Theorem (Cut-elimination)

The cut rules Cut and Cut are admissible in G3-HLJ<sub>S4</sub>

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We propose a term assignment system for the **G3-HLJ<sub>S4</sub>**,  $\lambda_{seq}^{\Box}$ , to get the computational model

As [Ohori '99] did for a G3-style prop. int. sequent calc., we assign terms to  $G3-HLJ_{S4} + Cut$  as follows:

Init/Right rules: assign  $\lambda$ -terms, as we do for N.D. system

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Left/Cut rules: assign the so-called "let expression"

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- Left/Cut rules: assign the so-called "let expression"

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Assign the modal  $\lambda$ -term [Pfenning+'01] to the init/right rules:

$$\overline{\Delta; \Gamma, x : A \vdash x : A} \xrightarrow{Ax} \overline{\Delta, u : A; \Gamma \vdash u : A} \square Ax$$

$$\underline{\Delta; \Gamma \vdash M : A} \xrightarrow{\Delta; \Gamma \vdash N : B} \land R$$

$$\underline{\Delta; \Gamma \vdash \langle M, N \rangle : A \land B} \xrightarrow{\Delta; \Gamma \vdash \lambda x : A . M : B} \supset R$$

$$\underline{\Delta; \Gamma \vdash \lambda x : A . M : A \supset B} \supset R$$

$$\underline{\Delta; \emptyset \vdash M : A} \square R$$

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$$\frac{\Delta; \Gamma, x : A \land B, y : A, z : B \vdash M : C}{\Delta; \Gamma, x : A \land B \vdash \mathsf{let} \langle y, z \rangle = x \mathsf{in} M : C} \land L$$

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The reduction intuitively proceeds, e.g., as:  $(\text{let } \langle y, z \rangle = \langle N, L \rangle \text{ in } M) \rightsquigarrow M[y := N, z := L]$ 

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The reduction intuitively proceeds, e.g., as: (let  $\langle y, z \rangle = \langle N, L \rangle$  in M)  $\rightsquigarrow M[y := N, z := L]$  The rules for the other left rules are defined similarly:

$$\frac{\Delta; \Gamma, x : A \supset B \vdash M : A}{\Delta; \Gamma, x : A \supset B, y : B \vdash N : C} \supset L$$

$$\frac{\Delta, u : A; \Gamma, x : \Box A \vdash M : B}{\Delta; \Gamma, x : \Box A \vdash \mathsf{let} \mathsf{ box } u = x \mathsf{ in } M : B} \Box L$$

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The term assignment for cut rules are defined as a "composition" of two constructions, again by using let-expressions:

$$\frac{\Delta; \Gamma \vdash M : A \qquad \Delta; \Gamma, x : A \vdash N : B}{\Delta; \Gamma \vdash \text{let } x = M \text{ in } N : B} \text{Cut}$$
$$\frac{\Delta; \emptyset \vdash M : A \qquad \Delta, u : A; \Gamma \vdash N : B}{\Delta; \Gamma \vdash \text{let } u = M \text{ in } N : B} \Box \text{Cut}$$

Let us consider the cut-elimination for conjunction:

$$\frac{\vdash M : A \quad \vdash N : B}{\vdash \langle M, N \rangle : A \land B} \land R \quad \frac{x : A \land B, y : A, z : B \vdash L : C}{x : A \land B \vdash \mathsf{let} \langle y, z \rangle = x \mathsf{in} L : C} \land L \\ \vdash \mathsf{let} x = \langle M, N \rangle \mathsf{in} \mathsf{let} \langle y, z \rangle = x \mathsf{in} L : C \qquad \mathsf{Cut}$$

To eliminate cuts, all we have to do is to compute:

$$(\textbf{let } x = \langle M, N \rangle \textbf{ in let } \langle y, z \rangle = x \textbf{ in } L)$$
  
 
$$\rightsquigarrow L[y := M, z := N, x := \langle M, N \rangle]$$

but we do not want to use "meta-level" substitution

Fortunatelly, the (local) cut-elimination step defined in the G3-style is exactly what we want!

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but we do not want to use "meta-level" substitution

Fortunatelly, the (local) cut-elimination step defined in the G3-style is exactly what we want!

(A part of) translation rules are obtained as follows: Optimization  $(\operatorname{let} x = M \operatorname{in} x) \rightsquigarrow M$   $(\operatorname{let} x = M \operatorname{in} y) \rightsquigarrow y$ 

Flattening

$$(\text{let } w = (\text{let } \langle y, z \rangle = x \text{ in } M) \text{ in } N) \rightsquigarrow (\text{let } \langle y, z \rangle = x \text{ in let } w = M \text{ in } N)$$
$$(\text{let } y = (\text{let box } u = x \text{ in } M) \text{ in } N) \rightsquigarrow (\text{let box } u = x \text{ in let } y = M \text{ in } N)$$

Decomposition

 $(\text{let } x = \langle M, N \rangle \text{ in let } \langle y, z \rangle = x \text{ in } L) \rightsquigarrow (\text{let } y = M \text{ in let } z = N \text{ in let } x = \langle y, z \rangle \text{ in } L)$  $(\text{let } x = \text{box } M \text{ in let box } u = x \text{ in } N) \rightsquigarrow (\text{let } u = M \text{ in let } x = \text{box } u \text{ in } N)$ 

These translation corresponds to "A-normal form compilation" in the theory of programming languages

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### Theorem (Subject reduction)

If  $\Delta$ ;  $\Gamma \vdash M : A$  and  $M \rightsquigarrow M'$ , then  $\Delta$ ;  $\Gamma \vdash M' : A$ 

Theorem (Strong normalization)

Every typable term is strongly normalizing

#### Corollary (Cut-elimination theorem)

 $\lambda_{seq}^{\square}$  enjoys the cut-elimination theorem, which also yields that every typable term can be reduced to the unique normal form

The following tells us that  $\lambda_{seq}^{\Box}$  can be used as a basis of model for the existing theory:

Theorem (Embedding from modal typed  $\lambda$ -calculus)

The modal  $\lambda$ -calc.  $\lambda^{\Box}$  [Pfenning+ '01] can be embedded into  $\lambda_{seq}^{\Box}$ : • If  $\Delta; \Gamma \vdash M : A$  in  $\lambda^{\Box}$ , then  $\Delta; \Gamma \vdash [M] : A$  in  $\lambda_{seq}^{\Box}$ • If  $M \rightsquigarrow M'$  in  $\lambda^{\Box}$ , then  $[M] \rightsquigarrow [M']$  in  $\lambda_{seq}^{\Box}$ where [-] means the translation mapping from  $\lambda^{\Box}$  to  $\lambda_{seq}^{\Box}$ 

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## Conclusion and future work

### Conclusion

- A cut-free higher-arity sequent calc. for intuitionistic S4: HLJ<sub>S4</sub> and G3-HLJ<sub>S4</sub>
- (A cut-free higher-arity sequent calc. for classical S4: HLK<sub>S4</sub> and G3-HLK<sub>S4</sub>)
- The corresponding term calculus for G3-HLJ<sub>S4</sub>
- Future work
  - The corresponding term calculus for the classical version, following the work of  $\lambda\mu$ -calculus for modal logic [Kimura+'11]
  - (Ongoing work with Akira Yoshimizu): Geometry of Interaction semantics for modal logic in terms of MELL, following the work of Gol semantics for PCF [Mackie '95]

# Appendix



# Cut-elimination (1)

$$let x = y in M \rightsquigarrow M[x := y]$$

$$let x = u in M \rightsquigarrow M[x := u]$$

$$let u = v in M \rightsquigarrow M[u := v]$$

$$let x = M in x \rightsquigarrow M$$

$$let x = M in y \rightsquigarrow y$$

$$let u = M in x \rightsquigarrow x$$

$$let u = M in v \rightsquigarrow v$$

$$let u = M in v \rightsquigarrow v$$

$$let x = M in N \mapsto u$$

$$let z = (let y = x M in N) in L \rightsquigarrow let y = x M in let z = N in L$$

$$let w = (let \langle y, z \rangle = x in M) in N \rightsquigarrow let \langle y, z \rangle = x in let w = M in N$$

$$let y = (let box u = x in M) in N \rightsquigarrow let box u = x in let y = M in N$$

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 $\begin{aligned} & |et x = L \text{ in } |et z = y \text{ } M \text{ in } N \rightsquigarrow |et z = y (|et x = L \text{ in } M) \text{ in } |et x = L \text{ in } N \\ & |et x = N \text{ in } |et \langle y, z \rangle = w \text{ in } M \rightsquigarrow |et \langle y, z \rangle = w \text{ in } |et x = N \text{ in } M \\ & |et x = L \text{ in } case w \text{ of } [y] \text{ } M \text{ or } [z] \text{ } N \rightsquigarrow case w \text{ of } [y] (|et x = L \text{ in } M) \text{ or } [z] (|et x = L \text{ in } N) \\ & |et x = N \text{ in } |et \text{ box } u = y \text{ in } M \rightsquigarrow |et \text{ box } u = y \text{ in } |et x = N \text{ in } M \\ & |et y = \lambda x : A.M \text{ in } |et z = y \text{ } N \text{ in } L \rightsquigarrow |et y = \lambda x : A.M \text{ in } |et x = N \text{ in } |et x = M \text{ in } L \\ & |et x = \langle M, N \rangle \text{ in } |et \langle y, z \rangle = x \text{ in } L \rightsquigarrow |et y = M \text{ in } |et x = u_1^{A \lor B}(y) \text{ in } N \\ & |et x = u_1^{A \lor B}(M) \text{ in } case x \text{ of } [y] \text{ } N \text{ or } [z] L \rightsquigarrow |et z = M \text{ in } |et x = u_1^{A \lor B}(z) \text{ in } N \\ & |et x = box \text{ } M \text{ in } |et box u = x \text{ in } N \rightsquigarrow |et u = M \text{ in } |et x = box u \text{ in } N \end{aligned}$ 

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