Relationship between Switchings and Introduction Rules of Multiplicative Connectives

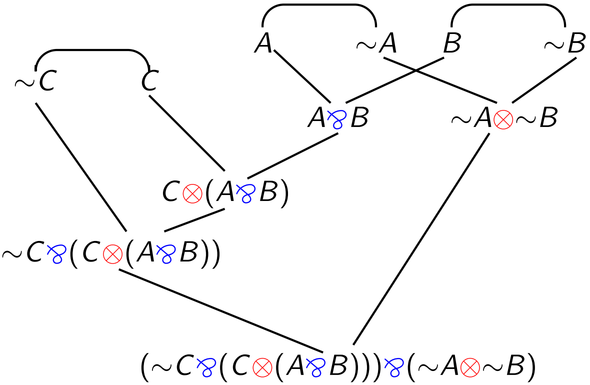
Yuki Nishimuta (Keio Univ.)

Second Workshop on Mathematical Logic and Its Application

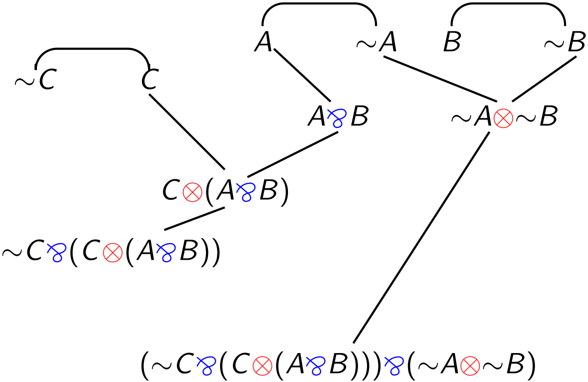
Background: proof-net

- Jean-Yves Girard introduced proof-net theory and gave a graphical characterization of sequent calculus proofs.
- Logical connectives defined on proof-nets are characterized by the notion of switching, which is a choice and deletion of one of two edges for each $\parr$-node.

An example of proof-net: before switching



Example of proof-net: after a switching



A difference between the conjunction $\otimes$ and the disjunction ⅋ is whether one of two edges is deleted or not.

Aim of this presentation

Problem

How are logical connectives defined on proof-nets related to those defined on sequent calculus?

- We address this problem by a generalization of logical connectives.
- Danos and Regnier (1989) generalized the notion of multiplicative connectives.
- We will extend the notion of proof-nets using these generalized multiplicative connectives and consider the notion of binary switching as the special case.

Aim of this presentation

Our approach for clarifying a relationship between inference rules and switchings is the following;

When inference rules of logical connectives are given in a sequent calculus, we construct switchings from inference rules.

Our result

Inference rules of logical connectives determine switchings (behavior of logical connectives on graphs) in the sense of our notion of (generalized) switching: partition switching.

Correspondence between a partition and a sequent [DR, 1989]

Danos and Regnier used the notion of partition to define generalized multiplicative connectives.

- We consider partitions of a natural number n.
- A partition consists of several classes $(-)$.
- A sequent $\vdash$ corresponds to a class $(-)$.
- When we consider a partition, we may omit contexts of an inference rule.

The principal formulas of upper sequents can be expressed as a partition of natural number.

$$\frac{\vdash A_1 \qquad \vdash A_2}{\vdash A_1 \otimes A_2}$$

$$p = \{(1)(2)\}$$

Correspondence between a partition and a sequent

When different formulas are in the same sequent, the corresponding numbers are contained in the same class.

$$\frac{\vdash\ A_1, A_2}{\vdash\ A_1 \parr A_2}$$

$$q = \{(1, 2)\}$$

Correspondence between a partition and a sequent

A_i is an atom (or meta variable).
Ex. 1: $(A_1 \otimes A_2) \parr A_3$.

$$\dfrac{\dfrac{\vdash A_1, A_3 \qquad \vdash A_2}{\vdash A_1 \otimes A_2, A_3}}{\vdash (A_1 \otimes A_2) \parr A_3} \qquad \dfrac{\dfrac{\vdash A_1 \qquad \vdash A_2, A_3}{\vdash A_1 \otimes A_2, A_3}}{\vdash (A_1 \otimes A_2) \parr A_3}$$

The formula $(A_1 \otimes A_2) \parr A_3$ corresponds to the set of partitions $P = \{p_1, p_2\}$: $p_1 = \{(1,3)(2)\}, p_2 = \{(2,3)(1)\}$.

Ex. 2: $(A_1 \parr A_2) \otimes (A_3 \parr A_4)$

$$\dfrac{\dfrac{\vdash A_1, A_2}{\vdash (A_1 \parr A_2)} \qquad \dfrac{\vdash A_3, A_4}{\vdash (A_3 \parr A_4)}}{\vdash (A_1 \parr A_2) \otimes (A_3 \parr A_4)}$$

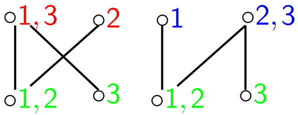
$$p = \{(1,2),(3,4)\}$$

Meeting graph [DR, 1989]

We can define the dual connective $\mathscr{C}^*$ of $\mathscr{C}$ by means of meeting graphs.

$\mathscr{C}(A,B,C) = (A \otimes B) \parr C$: $p_1 = \{(1,3)(2)\}, p_2 = \{(2,3)(1)\}$ $P = \{p_1, p_2\}$

$\mathscr{C}^*(\sim A, \sim B, \sim C) = (\sim A \parr \sim B) \otimes \sim C$: $q = \{\{1,2\}\{3\}\}$ $Q = \{q\}$



$$\frac{\vdash \Gamma, A_1, A_3 \qquad \vdash \Delta, A_2}{\vdash \Gamma, \Delta, \mathscr{C}(A_1, A_2, A_3)} \quad \frac{\vdash \Gamma, A_1 \qquad \vdash \Delta, A_2, A_3}{\vdash \Gamma, \Delta, \mathscr{C}(A_1, A_2, A_3)}$$

$$\frac{\vdash \Gamma, A_1, A_2 \qquad \vdash \Delta, A_3}{\vdash \Gamma, \Delta, \mathscr{C}^*(A_1, A_2, A_3)}$$

For given partitions p_1 and p_2, we can construct the partition q.

Cut-elimination for generalized connectives

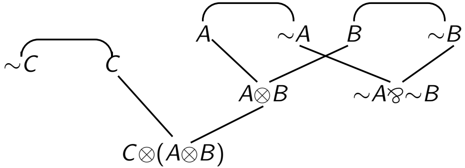
The definition of a generalized connective guarantees the main reduction step of the cut-elimination.

Fact 1

(Danos-Regnier, 1989, Lemma 1) The main reduction of the cut-elimination between a pair of generalized connectives $(\mathscr{C}, \mathscr{C}^*)$ *holds.*

Idea of Sequentialization Theorem

Sequentialization Theorem says that a graphically represented multiplicative proof-structure can be translated to a sequent calculus proof if a proof-structure satisfies some correctness criterion.

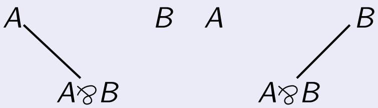


$$\dfrac{\dfrac{\vdash A, \sim A \qquad \vdash B, \sim B}{\vdash A \otimes B, \sim A, \sim B} \qquad \vdash C, \sim C}{\dfrac{\vdash C \otimes (A \otimes B), \sim A, \sim B, \sim C}{\vdash C \otimes (A \otimes B), \sim A \parr \sim B, \sim C}}$$

$\parr$-switching

Definition 1

We denote the set of $\parr$-links in a proof-structure $\mathscr{S}$ as $\parr(\mathscr{S})$. A switching l of a proof-structure $\mathscr{S}$ is a function $f : \parr(\mathscr{S}) \rightarrow \{left, right\}$. One of two edges of $\parr$ is deleted by a switching.



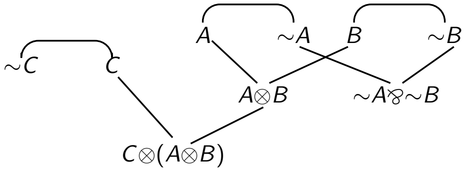
Definition 2 (DR, 1989)

If for any switching l, the induced graph $\mathscr{S}_l$ is connected and acyclic, then a proof-structure $\mathscr{S}$ is correct. A proof-net $\mathscr{S}$ is a correct proof-structure.

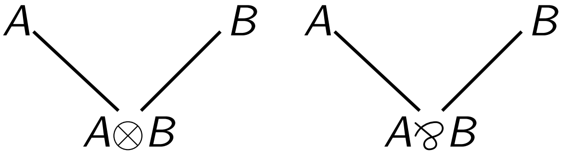
Sequentialization theorem (binary case)

Theorem 1

(Seqentialization of MLL*) (Girard, 1987; Danos and Regnier, 1989)*
If $\mathscr{S}$ is a proof-net, then a proof-structure $\mathscr{S}$ is sequentializable.

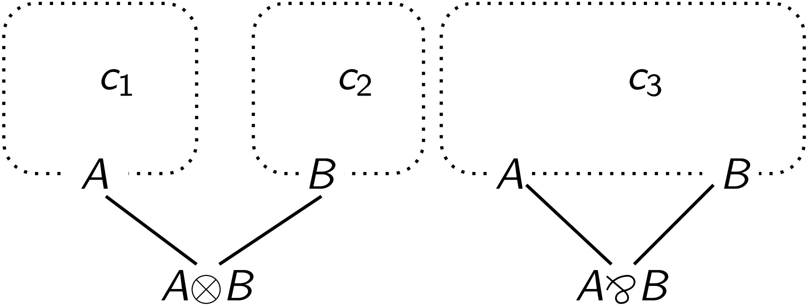


$$\dfrac{\dfrac{\dfrac{\vdash A,\sim A \qquad \vdash B,\sim B}{\vdash A\otimes B,\sim A,\sim B} \qquad \vdash C,\sim C}{\vdash C\otimes(A\otimes B),\sim A,\sim B,\sim C}}{\vdash C\otimes(A\otimes B),\sim A\parr\sim B,\sim C}$$



Explicitly, information about logical connectives on graphs is

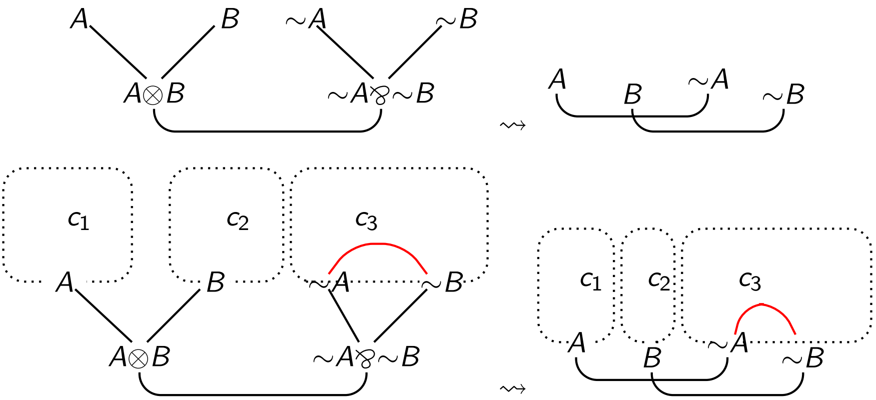
Switching



Information about logical connectives on graphs is

Switching + Connected component

Cut-elimination in MLL-proof-structure



Cut-elimination on graphs is related to information about connected components.

Idea of our switching

Idea of our switching

Our partition switching chooses exactly one element from each class of a given partition.

$$\frac{\vdash\ A_1 \qquad \vdash\ A_2}{\vdash\ A_1 \otimes A_2}$$

$$p = \{(1)(2)\}$$

$$\frac{\vdash\ A_1, A_2}{\vdash\ A_1 \parr A_2}$$

$$q = \{(1, 2)\}$$

Our partition switching

Definition 3

Let $\mathscr{S}$ be a proof-structure containing $\mathbb{C}$-links $\mathscr{C}_i$ $(i = 1, \ldots, n)$. A partition switching I of $\mathscr{S}$ is a function f such that for any i and some $p \in P_{\mathscr{C}}$ (where $p = \{(p_{11}, \ldots, p_{1m_1}), \ldots, (p_{k1}, \ldots, p_{km_k})\}$, $p_{jk} \in \{1, \ldots, n\}$), f selects one element from each class of p; $f(p) = \{p_{1f(1)}, \ldots, p_{kf(k)}\}$ where $p_{if_i} \in Class(p)$.

例: $\mathscr{C}(A_1, A_2, A_3) = (A_1 \otimes A_2) \parr A_3$

$$p_1 = \{(a_1, a_3), (a_2)\} \Longrightarrow_{I_1} a_1, a_2$$
$$p_1 = \{(a_1, a_3), (a_2)\} \Longrightarrow_{I_2} a_3, a_2$$

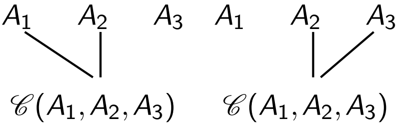
or

$$p_2 = \{(a_1), (a_2, a_3)\} \Longrightarrow_{J_1} a_1, a_2$$
$$p_2 = \{(a_1), (a_2, a_3)\} \Longrightarrow_{J_2} a_1, a_3$$

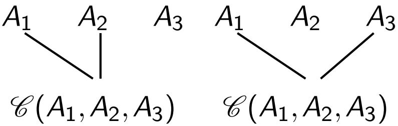
Partition switching

$$p_1 = \{(a_1, a_3), (a_2)\} \Longrightarrow_{I_1} a_1, a_2 \;\; p_1 = \{(a_1, a_3), (a_2)\} \Longrightarrow_{I_2} a_3, a_2$$ or

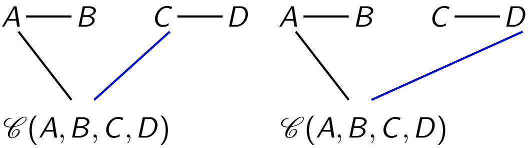
$$p_2 = \{(a_1), (a_2, a_3)\} \Longrightarrow_{J_1} a_1, a_2 \;\; p_2 = \{(a_1), (a_2, a_3)\} \Longrightarrow_{J_2} a_1, a_3$$



or



Partition switching



etc.

We connect formulas in the same class (sequent). After that, we connect $\mathscr{C}$-node and exactly one upper node for each class.

$$\frac{\vdash \Gamma_1, A, B \qquad \vdash \Gamma_2, C, D}{\vdash \Gamma_1, \Gamma_2, \mathscr{C}(A, B, C, D)}$$

Information about partitions (inference rules) is the almost same as information about connected component on graphs.

Main result: generalization of Sequentialization Theorem

Proposition 1

Let $\mathscr{S}$ be an arbitrary proof-structure containing arbitrary $\mathbb{C}$-links $\mathscr{C}_i$ $(i = 1, \ldots, n)$. If $\mathscr{S}$ is a proof-net in the sense of the partition switching, then $\mathscr{S}$ is sequentializable.

Our proof of this theorem is similar as Olivier Laurent's proof of Sequentialization Theorem (Laurent, 2003).

Conclusion

- The main reduction of the cut-elimination on graphs is related to information about connected components.
- Information about connected components is almost the same as information of inference rules.
- We can obtain switchings (behaviors on graphs) from inference rules via our partition switching.

Cut-Elimination on graphs⇒ Information about connected component ⇒ Information about inference rules ⇒ Information about switchings

Informally, this result says that graphical view of logical connectives and inferential view of these are not so different in some sense.

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