Monday, 8 March

9:00 Peter Aczel
Some remarks on axiom systems for constructive set theory

9:45 SATO Kentaro
A Reverse Mathematics for Feasibility

10:15 **Coffee break (30 min)**

10:45 Francesco Ciraulo (j.w.w. Giovanni Sambin)
A Galois connection between basic covers and binary positivities

11:30 Simon Kramer
An Intuitionistic Foundation for Interactive Computation

12:00 **Lunch (2 h)**

14:00 Erik Palmgren
Sets, setoids and groupoids

14:45 Anton Hedin
Locally Scott formal topologies and constructive interval analysis

15:30 **Coffee break (30 min)**

16:00 Danko Ilik
Constructive Completeness Theorems and Delimited Control

Tuesday, 9 March

9:00 Douglas S. Bridges
A return to apartness spaces

9:45 Josef Berger
Aligning WKL, UC, *and* FAN

10:15 **Coffee break (30 min)**

10:45 Hannes Diener
A constructive look at the Vitali Covering Theorem

11:30 Keita Yokoyama
Comparing non-standard axioms with axioms of second-order arithmetic

12:00 **Lunch (2 h)**

14:00 Maria Emilia Maietti
The role of unique choice in the minimalist foundation

14:45 Giovanni Curi
Intuitionistic theorems that fail constructively

15:30 **Coffee break (30 min)**

16:00 Laura Crosilla
On constructive operational set theory

Wednesday, 10 March

9:00 Susumu Hayashi
Tanabe's Logic of Species and Brouwer's Theory of Continuum
—Japanese philosophy and intuitionistic mathematics—

9:45 Takuro Onishi
Proof-Theoretic Semantics and Constructivity

10:15 **Coffee break (30 min)**

10:45 Mariko Yasugi and Masako Washihara
On sequential computability of a function

11:30 Yoshihiro Maruyama
Another Notion of Point in Locale Theory

18:00 **Conference dinner**

Thursday, 11 March

9:00 Klaus Keimel
Choquet type theorems and continuous domains

9:45 Shouhei Izawa
Some existence axioms in finite order arithmetic

10:15 **Coffee break (30 min)**

10:45 Hideki Tsuiki
Domain Representations Derived from Dyadic Subbases

11:30 Takayuki Kihara
Degrees of co-c.e. closed sets with specific computability-theoretic properties

12:00 **Lunch (2 h)**

14:00 Jan von Plato
Geometric proof theory of intuitionistic geometry

14:45 Andrej Bauer
On the Bourbaki-Witt and Knaster-Tarski fixed-point theorems

15:30 **Coffee break (30 min)**

16:00 Vasco Brattka
Non-determinism and choice in computable reverse analysis

Friday, 12 March

9:00 Simon Huber, Basil Karadais and Helmut Schwichtenberg
Towards a formal theory of computability

9:45 Ryota Akiyoshi
An Interpretation of Takeuti's Reduction

10:15 **Coffee break (30 min)**

10:45 Wim Veldman
The principle of open induction and some of its equivalents

11:30 Yoichi Hirai
An Intuitionistic Epistemic Logic

12:00 **Lunch (2 h)**

14:00 Peter Schuster
On Paths and Points

14:45 Michael Rathjen
Conservation and independence results in intuitionistic set theory

- PETER ACZEL, *Some remarks on axiom systems for constructive set theory*
  University of Manchester

  Perhaps the most frequently mentioned axiom systems for Constructive Set Theory (CST) have been CZF and CZF$^+$(= CZF + REA). But there are a variety of other axiom systems for CST that may also be of interest. One may be interested to find the weakest natural axiom system for the development of a certain topic such as finitary inductive definitions or deterministic inductive definitions. One may be interested to determine which axiom systems for CST have the set existence property? The fully impredicative axiom system IZF does not have the set existence property. But the axiom system IZF$_R$, obtained from IZF by using the Replacement Scheme instead of Collection does have the set existence property, an old result of Myhill and Friedman. My talk will review a selection of these and other topics concerning CST.

- RYOTA AKIYOSHI, *An Interpretation of Takeuti's Reduction*
  Keio University

  The first constructive consistency proof for an impredicative subsystem of analysis has been given by G. Takeuti. He introduced a very complicated ordinal notations system called "ordinal diagram" for the termination proof of his reduction steps. His proof is very difficult to follow because of the complexity of ordinal diagrams.

  After Takeuti's works, Buchholz introduced a new infinitary rule called "Ω-rule" for ordinal analysis for iterated inductive definitions. In order for that, he proves an impredicative cut-elimination theorem called the "collapsing theorem". Recently Buchholz explained Takeuti's reduction in terms of his Ω-rule.

  In this talk, we present an additional correspondence between Takeuti's reduction and Buchholz's one by examining Buchholz's work. Especially, we try to give an answer to the question why the transformed derivation in Takeuti's impredicative cut-elimination is "simpler" than the original derivation. This work is joint one with G. Mints.

- ANDREJ BAUER, *On the Bourbaki-Witt and Knaster-Tarski fixed-point theorems*
  University of Ljubljana

  The Bourbaki-Witt theorem states that a progressive map on a chain-complete poset has a fixed point. The (chain-complete version of the) Knaster-Tarski theorem that a monotone map on a chain-complete poset has a fixed point. In the talk I shall discuss the validity and failure of the theorems from a constructive point of view. While the theorems do not hold constructively, it is still interesting to study the circumstances under which they fail or hold. The work has been done jointly with Peter LeFanu Lumsdaine.

- JOSEF BERGER, *Aligning* WKL, UC*, and* FAN
  Ludwig-Maximilians-Universität München

  Working in the framework of Bishop's constructive mathematics, we want to compare the weak Konig lemma (WKL), the uniform continuity theorem (UC) for integer-valued functions on Cantor space, and Brouwer's fan theorem (FAN) for decidable bars. We know that WKL implies UC in HA_ω and therefore in Bishop's constructive mathematics. Furthermore, it is easy to see that UC implies FAN. We aim at formulating the three axioms in a uniform way. We hope that this will improve our understanding of the differences between them.

- VASCO BRATTKA, *Non-determinism and choice in computable reverse analysis*
  University of Cape Town

  We report on some recent results in computable reverse analysis. We use the Weihrauch lattice to classify operations according to their degree of non-computability. This approach also allows to classify the degree of non-computability of theorems such as the Baire Category Theorem or Weak Koenig's Lemma. It turns out that choice of particular spaces characterizes certain classes of computable functions (including the equivalence classes of the above theorems) and using a close relation to non-deterministically computable functions we can simplify and unify proofs of some known results.

This talk is based on joint work with Guido Gherardi, on the one hand, and Matthew de Brecht and Arno Pauly, on the other hand.

- Douglas S. Bridges, *A return to apartness spaces*
  University of Canterbury

  I shall outline the latest, and we hope final, axiomatisation of apartness spaces, and some key ideas from the theory. The lectures will be suitable for postgraduate students as well as faculty.

- Francesco Ciraulo (j.w.w. Giovanni Sambin), *A Galois connection between basic covers and binary positivities*
  University of Palermo

  The introduction (see [4] and [5]) of the binary positivity predicate in the definition of a formal topology has generated the need for understanding its interplay with the cover relation. In particular, it seems interesting to answer the question whether and to what extent the binary positivity is determined by the cover (or vice versa). The joint work I present goes exactly towards this direction. The framework we chose for our analysis is that of *basic* topologies (the cover relation needs to satisfy only reflexivity and transitivity; the positivity enjoys *co-reflexivity* and *co-transitivity*, in a precise sense, and is linked to the cover by a certain *compatibility* condition).
  I shall show that the binary positivity is completely determined by the cover if and only if the latter can be generated inductively (in that case the positivity is defined by coinduction). Though very important, the class of inductive(-coinductive) topologies does not include all known examples (think of the Dedekind-MacNeille cover on a Heyting algebra; see [2] and [3]). In fact, another important class exists, that of pointwise definable or *representable* topologies (see [4] and [5]). In that case, and only in that case, it is the cover relation to be completely determined by the binary positivity (dually to what happens for generated topologies). The class of representable topologies can neither include (unless Markov's principle

holds) nor be included in the class of generated ones (provided that one wants to avoid the axiom of choice, such as in Maietti-Sambin's minimalist foundation). By the way, there are examples of basic topologies that are neither generable nor representable.
All this can be explained and expressed categorically (though impredicatively) in the form of a (contravariant) Galois connection (or dual adjunction) between all basic covers and binary positivities on the same set.

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- Laura Crosilla, *On constructive operational set theory*
  University of Leeds

Constructive operational set theory is a constructive theory of sets which incorporates a notion of partial application. It can be seen as a bridge between two distinct traditions within the foundations for constructive mathematics: Aczel's constructive Zermelo Fraenkel set theory and Feferman's explicit mathematics. More precisely, it is a constructive version of Feferman's classical operational set theory and of Beeson's impredicative intuitionistic set theory with rules. In constructive operational set theory a notion of operation or rule can be described, aside the notion of constructive set. While sets are fully extensional, operations are non-extensional and partial.

We shall here present work in progress on systems of constructive set theory with operations.

- Givanni Curi, *Intuitionistic theorems that fail constructively*
  Verona University

  It may be argued that a reasonable necessary condition to qualify a given mathematical result as constructive is that it can be formulated within some extension of constructive set theory (CZF), or of type theory (CTT), that is compatible with the generalized uniformity principle (GUP). The fully impredicative intuitionistic set theory IZF is an example of an extension of CZF that is not consistent with GUP. On the other hand, since CZF extended by (REA, PA and) the impredicative unrestricted separation scheme is compatible with GUP, the considered necessary condition for constructivity is clearly not a too severe one. In this talk I will describe some example of results that can be derived in topos logic and in intuitionistic set theory IZF, but are non-constructive according to the present criterion.

- Hannes Diener, *A constructive look at the Vitali Covering Theorem*
  Universität Siegen

  This is joint work with Anton Hedin.

  We will present work investigating the Vitali Covering Lemma (VCL) from various constructive angles. A Vitali Cover of a metric space is an open cover such that for every point there exists an arbitrarily small set of the cover covering this point. The VCL now states, that for any Vitali Cover one can find a finite, disjoint family of sets in the Vitali Cover that cover the entire space up to a set of a given measure. Being about extracting a sub-family from a given cover with certain properties, the VCL is superficially similar to the Heine-Borel Theorem. The key difference is that in the VCL one is interested in extracting a *disjoint* family of sets. The VCL is often used in classical measure theory. Thus, apart from being interesting in itself, a constructive analysis of the VCL is of interest for doing measure theory constructively. In this talk, among other things, we will show where it can be located within the hierarchy of constructive reverse mathematics, and we will show how to construct a recursive counterexample.

- SUSUMU HAYASHI, *Tanabe's Logic of Species and Brouwer's Theory of Continuum —Japanese philosophy and intuitionistic mathematics—*
  Kyoto University

  Hajime Tanabe (田辺元, 1885-1962) is the first philosopher of science in Japan. His books on sciences and mathematics were quite popular among high school and university students before WW2. His writings contributed much to popularize the foundations of mathematics in Japan in 1910-20's.. He later shifted to the more traditional areas of philosophy and developed a theory of **Logic of Species** (種の論理) in 1930-40's. It is a traditional logic in the sense of Aristotle and German idealism. In Aristotelian and Hegelian logic, species are merely subsets of classes, and individuals are the smallest species, and classes are collections of individuals. The notion of species has a unique place in Tanabe's logic. Logic of Species was intended as a social philosophy to rival Marxism prevailing then among Japanese young intelligentsia, especially students of Kyoto University, where he taught. Species in his sense are racial groups, racially homogeneous societies or nations. It was, however, formulated as a general logic applicable also to natural sciences and mathematics. It is both scientific and political as some Neo-Kantian philosophy, e.g., Hermann Cohen's philosophy of infinitesimals. Several key conditions were posed for Logic of Species. Species are not merely collections of independent individuals. For Tanabe and most members of Kyoto School (of philosophy), it was a shared assumption that democratic western societies and cultures were in the process of a final decline and they and their basic assumption, individualism, must be overcome. Thus, species comes first rather than individual. Nonetheless, another condition was the liberty of individuals. Logic of Species is a kind of dialectics. A species must not be a mere set of individuals, but each individual must be "attached" into it in some sense. Nonetheless, an individual has liberty to oppose to its species so that its opposition or the conflict (contradiction) between these two entities brings "Aufhebung". Tanabe, who learned Brouwer's theory of choice sequences through writings by Oskar Becker, Fraenkel and Heyting, found Brouwerian theory of continuum a good model of Logic of Species.

- Anton Hedin, *Locally Scott formal topologies and constructive interval analysis*
  Uppsala University

  We show how to embed certain formal topologies in locally Scott formal topologies. In [3] it is shown that any continuous domain can be represented by the formal space of a (stable) locally Scott formal topology. The embedding can then be seen as a domain representation of the original space. We also prove a lifting result for morphisms. The prime example of a (stable) locally Scott formal topology is the topology of partial reals [3]. Classically its space consists of closed intervals (a similar extension of the reals is given in [6]). We show that partial reals that define compact overt subspaces of the real numbers correspond precisely to closed intervals with constructible endpoints. The lifting of morphisms in this case give tight extensions of continuous real valued functions in the sense of interval analysis. We use these results for a constructive approach to the domain theoretical framework for real analysis as developed by Edalat et al. [2].

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- Yoichi Hirai, *An Intuitionistic Epistemic Logic*
University of Tokyo

  Under a fixed countable set of agents, we consider extending the language of propositional logic with a modality K_a for each agent a and adding one line to Brouwer-Heyting-Kolmogorov interpretation of logical formulas. Namely, a proof of $K_a\varphi$ is agent a's acknowledgement of a proof of φ. Aiming for simplicity, we make assumptions on agents and acknowledgements and formalize the assumptions in a deduction system. The system is so strong that the modalities lose their meanings if we allow double negation elimination. For a Kripke semantics, we show soundness and strong completeness of the system. We discuss the relashionship with classical epistemic logic and show that the set of theorems is not recursive when there are more than two agents.

- Danko Ilik, *Constructive Completeness Theorems and Delimited Control*
INRIA

  Motivated by facilitating reasoning about meta-theory inside the Coq proof assistant, we revisit constructive proofs of completeness theorems.

  In the first part of the talk, we discuss Krivine's constructive proof of completeness of classical logic and its computational contents.

Then we present a class of Kripke-style models for which completeness can be shown via a normalisation-by-evaluation proof, and we compare its computational behaviour to the one of Krivine's proof.

In the second part, we revisit constructive completeness of intuitionistic logic (with disjunction and the existential) w.r.t. Kripke models. We discuss Danvy's Ocaml program for normalisation-by-evaluation of simply-typed lambda calculus with sums, and delimited control operators therein used. From his program, we extract a notion of model (which has the same shape as the notion presented in part 1 of the talk) that is sound and complete for full intuitionistic predicate logic. Finally, we present the cube of Herbelin and discuss some potential gains of adding delimited control operators to an intuitionistic or classical proof system.

- Shouhei Izawa, *Some existence axioms in finite order arithmetic*
  Tohoku University

  There are many formal systems for formalization of mathematics such as set theory, second order arithmetic etc. My talk is related to one kind of formal system, finite order arithmetic. The system finite order arithmetic is based on lambda calculus on infinitely many sorted language, where the sorts that system has is 0 (0 corresponded to the set of all natural numbers) and obtained from 0 by finite iteration of the construction of a new sort from given two sorts (it corresponds to the set of all maps from one sort to the other sort). In this system, many axioms, such as comprehension, choice, (transfinite) recursion etc. are considerable in each sort. I talk about comparisons of strength of these axioms.

- Klaus Keimel, *Choquet type theorems and continuous domains*
  Technische Universität Darmstadt

  We will deal with theorems often called Choquet - Matheron - Kendall type theorems. They are a basic tool in the theory of random sets (see, e.g., [M]). One can view these theorems as representing probability measures or more general measures by distribution functions (see [R]) thus generalizing a basic fact about measures on the real line.

While the classical theorems deal with locally compact Hausdorff spaces often supposed to be second countable we will extend these the theory to a non Hausdorff setting as occurring in domain theory.

It is our claim that continuous domains in the sense of D.S. Scott (see, e.g., [GHKLMS]) yield a natural framework for Choquet - Matheron - Kendall type theorems. In the classical case these theorems are mostly applied to spaces of closed or compact subsets of locally compact Hausdorff spaces with the Fell and the Vietoris topology, respectively. In our domain theoretical setting we can apply these theorems to the representation of (probability) measures on powerdomains as the Hoare and the Smyth powerdomain of closed and compact saturated subsets, respectively, over a class of T_0-spaces containing all locally compact Hausdorff spaces as well as all continuous domains with the Scott topology.

We report on ongoing work in collaboration with Jean Goubault-Larrecq who has initiated this strand of research and had obtained the basic results.

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- Takayuki Kihara, *Degrees of co-c.e. closed sets with specific computability-theoretic properties*
  Tohoku University

We investigate specific computability-theoretic properties of co-computably enumerable closed sets in Cantor space. For example, we have our sights set on (Post's) immunity for closed sets introduced by Cenzer-Weber-Wu, and K-triviality (in the sense of algorithmic randomness) for closed sets introduced by Barmpalias-Cenzer-Remmel-Weber. Then we give results concerning the Medvedev/Muchnik degrees of such closed sets.

- SIMON KRAMER, *An Intuitionistic Foundation for Interactive Computation*
University of Tsukuba

Abstract We propose an intuitionistic foundation for interactive computation via a Curry-Howard isomorphism from interactive intuitionistic logic defined via a classical modal logic of interactive proofs. Our proposal is an interactive analog of the Gödel-Kolmogorov-Artëmov definition of intuitionistic logic as embedded into a classical modal logic of proofs, and of the Curry-Howard isomorphism between intuitionistic proofs and typed programs. That is, terms viewed as proofs are descriptions of constructive deductions, terms viewed as programs are prescriptions for computations, formulas viewed as propositions are proof goals, and formulas viewed as types are program properties. To agents, interactive proofs are message terms that induce the knowledge of their proof goal with their intended interpreters, and interactive computations are message communications between distributed interlocutors that compute that knowledge from the meaning of the communicated messages.

Keywords Curry-Howard isomorphism, interactive computation, intuitionistic logic, modal logics of knowledge and proofs, multi-agent systems.

Motivation In [3], interactive computation is proposed as the new paradigm of computation, as opposed to the old paradigm of non-interactive computation in the sense of the old sages like Turing and others. The motivation for this paper is the consensus of the contributors to [3], which is that the purpose of interactive computation ultimately is not the computation of result values, to which we consent, but the possibly unending interaction *itself*, from which we dissent. The interaction may well be unending, but it cannot be a self-purpose because if it were then all interactive programs would be quines—rhetorically exaggerated. (A quine is a program that produces its own and only that code [1].)

Goal Our goal is to reach consensus with the reader that values are only the means—not the ends—of interactive computation, and that **the purpose of interactive computation is *interpreted communication* between distributed agents—humans and/or machines—interacting via message passing.** (A communication channel/medium can be modelled as an agent in the sense of a machine.)

Problem So what is interpreted communication? According to Shannon [4]:

> The fundamental problem of [*un*interpreted] communication is that of reproducing at one point either exactly or approximately a message selected at another point.

In analogy, we declare:

> The fundamental problem of *interpreted* communication is that of [re]producing at one point either exactly or approximately the *intended meaning* of a message selected at another point.

Note that due to the distribution of the different agents in a communication system, which may have different views of the system, the agents constitute different message *interpretation contexts*. Hence, identical messages may well be interpreted differently in different contexts, and thus have different meanings to different agents. As a matter of fact, message misinterpretations are ubiquitous in human and/or machine communications, e.g., in communication protocols [2, Chapter 3], and may have serious or even catastrophic consequences, e.g., in the context of nuclear command and control [2, Chapter 13]. Indeed, [re]producing intended message meaning across interpretation contexts is a highly critical and non-trivial problem. But what does message meaning mean more precisely?

We argue that the meaning of a message in a given interpretation context is the *propositional knowledge* which the *individual knowledge* of that message induces in that context. By individual knowledge we mean knowledge in the sense of the transitive use of the verb "to know", here to know a message, such as the plaintext of an encrypted message. Whereas by propositional knowledge we mean knowledge in the sense of the use of the verb "to know" with a clause, here to know that a statement is true, such as that the plaintext of an encrypted message is (individually) unknown to potential adversaries.

Hence, an agent-centric paraphrase of our previous problem statement is:

> The fundamental problem of communication is that of inducing at one point either an intended knowledge or an intended belief with a message selected at another point.

With this paper, we shall confine us to inducing (necessarily true) knowledge, and leave induction of (possibly false) belief for further work. Here, interactive computations compute propositional *knowledge* (e.g., that the goal of this paper has been achieved), and they do so by passing as messages pieces of interactively or non-interactively computed individual knowledge (e.g., this paper). Again, result values are only the means—not the ends—of interactive computations.

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- Maria Emilia Maietti, *The role of unique choice in the minimalist foundation*
  University of Padua

In [1] we argued for the need of a minimalist foundation where the notion of function as functional graph is distinct from that of function as operation. We expressed this by saying that the axiom of unique turning a functional relation into an operation is not necessarily valid.

The main benefit of avoiding unique choice is that our foundation developed in [2] can be proved to be consistent with Brouwer's Bar Induction and formal Church thesis for operations.

After briefly reviewing the consistency proof [3] we talk about different topological structures behind Cantor space and related tree topologies in our foundation and in its extensions with unique choice or Brouwer's principles.

Joint work with G. Sambin

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- Yoshihiro Maruyama, *Another Notion of Point in Locale Theory*
  Kyoto University

  We consider another notion of point in locale theory and show a categorical duality based on that notion of point. Usually, a point of a frame is defined as a completely prime filter of the frame (see [2]). However, the points of many classical algebraic varieties equipped with the Zariski topologies (see [1]) cannot be recovered from the frames of their open sets by taking completely prime filter as point. Nevertheless, they can easily be recovered by taking maximal join-complete ideal as point. Our aim is to show a categorical duality based on the view that point is maximal join-complete ideal.

  There is a duality (sometimes called Isbell duality) between the category of spatial frames and the category of sober spaces (see [2]). Isbell duality is based on the view that point is completely prime filter. Thus the points of many classical algebraic varieties cannot be recovered via Isbell duality.

  In this talk we show that the view "point is maximal join-complete ideal" provides a different duality between the category of "m-spatial" frames and the category of T_1 spaces. Our duality makes it possible to recover the points of classical algebraic varieties from the frames of their open sets. We provide an algebraic characterization of m-spatiality. By our result, we may consider that the appropriate notion of point in locale theory cannot be uniquely determined and that there are two kinds of dualities (i.e., Isbell and our dualities) which are naturally induced by the two notions of point.

  We also consider applications of our method to domain theory, convexity theory and measure theory.

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- Takuro Onishi, *Proof-Theoretic Semantics and Constructivity*
  (Kyoto University/ JSPS)

In his *Logical Basis of Metaphysics* (1991), Dummett developed a proof-theoretic approach to logical semantics and argued that the inference rules of natural deduction of intuitionistic logic fully determine the meanings of logical constants they govern, while those of classical logic cannot be seen as such. His argument was based on constructivity of intuitionistic logic, especially on the fact that it enjoys the disjunction (and existence) property, but classical logic does not.

In his framework, a set of inference rules (introduction and elimination rules) concerning a logical constant is said to determine the meaning of the constant if (1) the introduction can be inverted, and (2) in his terminology, "harmony" and "stability" obtain between the introduction and the elimination, that is, the inverse of the introduction and the elimination are equivalent. Admittedly disjunction (and existential quantifier) in classical logic fails to satisfy (1). But it is arguable that (2), in particular, stability of the rules for disjunction (and existential quantifier) fails, or at least cannot be shown to hold by ordinary proof-theoretic means, even in intuitionistic logic. Something is wrong in his argument.

My diagnosis is that Dummett's appeal to constructivity is misplaced and makes his own conceptual framework of proof-theoretic semantics unclear. Proposing some revisions of assumptions he made, I will present a clearer view about when and why inference rules can be seen as "meaning conferring", which will also shed some light upon the nature of logical constants (cf. [2] and [3]). Finally I will discuss the relationship between constructivity of proofs and the role which inversion of introduction rules plays in the context of proof-theoretic semantics.

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- Erik Palmgren, *Sets, setoids and groupoids*
Uppsala University

  When formalising constructive mathematics in Martin-Lof type theory it is standard to use setoids (types with equivalence relations) to interpret sets. The notion of preset is well captured by a type equipped with its identity type as equivalence relation. However when it comes to families of sets it seems that there is a choice between proof relevant index sets (groupoids) and proof irrelevant index set (setoids). We aim to discuss these issues and the relation to the groupoid model of type theory by Hofmann and Streicher.

- Jan von Plato, *Geometric proof theory of intuitionistic geometry*
University of Helsinki

  Intuitionistic geometry seems to have been the first occasion when intuitionistic mathematics was studied axiomatically. It was the topic of Heyting's (1925) thesis (*cf.* also Heyting 1927). He used as basic notions apartness of points and of lines, and incidence of a point on a line. The last was an error, changed into "outsideness" in von Plato (1995) as dictated by the basic insight about "finitely precise" or computable basic notions in intuitionism.

  Following the program of *proof analysis* (Negri and von Plato 1998, Negri 2003), it is possible to convert the axioms of intuitionistic geometry into a system of rules that extends a suitable proof-theoretical calculus. Typical applications include *word problems:* Systems of proof search that decide whether an atomic formula, or a disjunctively interpreted collection of such, is derivable from given atomic formulas used as assumptions. Such results were given in von Plato (2010) for classical plane projective and affine geometries.

  Proof analysis becomes a very hard task combinatorially when the axioms are many. It is shown here how to axiomatize intuitionistic plane projective and affine geometries with just one basic notion and to define point and line equalities. As a result, the corresponding proof system has much fewer production rules. They use eigenvariables, similarly to Negri's (2003) method of conversion of geometric implications into rules. Results analogous to those in von Plato (2010) can now be proved rather easily.

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- Michael Rathjen, *Conservation and independence results in intuitionistic set theory*
  University of Leeds

  The first part of the talk will be concerned with constructing models for intuitionistic set theories in which various forms of the axiom of choice do not hold, utilizing techniques such as Lifschitz realizability and realizability permutation models. In the second part I would like to discuss Goodman-style conservation results for set theories which can be obtained by a combination of relativized realizability and forcing.

- SATO Kentaro, *A Reverse Mathematics for Feasibility*
  Kobe University

The speaker has been working on a research project [2, 3, 4], aiming to clarify the utility of "graphic conception of sets". Roughly speaking, by representing sets as directed graphs, we can interpret set theories within more general classes of systems and, therefore, we can introduce set theories that are conservative over various systems. This enables us to use set theory as a unified framework for a wide variety of mathematical systems (from feasible mathematics to large cardinal theories), and for several kinds of reverse mathematics.

The introduced set theories are characterized by Axiom β (which asserts the existence of transitive collapses of well-founded directed graphs) and behave quite differently from those set theories based on "iterative conception of sets" (including Kripke-Platek set theory and constructive set theory). Whereas the latter class of set theories has been studied extensively, graph-representation of sets does not seem to have been studied enough in spite of its simplicity (except few works, e.g., $\mathbf{ATR}_0^{\mathrm{set}}$ from [5, Chapter VII]).

As a part of this project, the speaker has enhanced *Bounded Reverse Mathematics* (BRM) programme, proposed by Cook and Nguyen [1].

This programme is aiming to identify the necessary and sufficient computable complexity to solve a given (finite-combinatorial) problem. Because their base theory $\mathbf{V}^0$ is associated with the complexity class $\mathbf{AC}^0$, results in this programme can be seen as $\mathbf{AC}^0$-reducibility (actually, in many cases, $\mathbf{AC}^0$-many-one-reducibility) results. One can easily find an analogy between systems used in BRM and those used in Friedman-Simpson's "orthodox" reverse mathematics, according to which bounded (set and number) quantifiers correspond to unbounded (set and number) quantifiers. However, Cook's base theory $\mathbf{V}^0$ corresponds to $\mathbf{ACA}_0$ (according to the analogy), not $\mathbf{RCA}_0$.

In this talk, a new theory $\mathbf{V}^-$ corresponding to $\mathbf{RCA}_0$ is introduced and several reverse-mathematical results based on this weaker system are shown, with comparison with those results in the "orthodox" RM. Moreover, this comparison will be sharpened by interpreted set theories because of uniformity. Several problems induced by $\mathbf{V}^-$ will also be discussed, e.g.: Despite such utility as a base theory, the associated complexity does not seem to fit in any reasonably defined computational models (introduced so far).

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[2] K. Sato, The strength of extensionality I: weak weak set theories with infinity, *Ann. Pure App. Logic* 157 (2009) 234-268.

[3] K. Sato, The strength of extensionality II: weak weak set theories without infinity, submitted.

[4] K. Sato, The strength of extensionality III: weak weak set theories with higher infinite, in preparation.

[5] S. Simpson, *Subsystems of Second Order Arithmetic*, Springer 1999.

- PETER SCHUSTER, *On Paths and Points*
University of Leeds

The infinite paths of a tree are the points of the space it represents. In a nutshell this well-known fact contains a large portion of the foundational issues that gave birth to constructive mathematics: excluded middle, choice, and impredicativity. We review how all three issues can be avoided at the same time whenever there is, in an appropriate sense, uniformly at most one infinite path.

More precisely, we show that for every detachable binary tree with uniformly at most one infinite path there is a spread which is an infinite path whenever the given tree is infinite. Here infinite paths are understood as formal points, i.e. as linear spreads, rather than as infinite sequences. In particular, this variant of the Weak Koenig Lemma allows for a constructive proof without unique choice. As a complement, the essential uniform uniqueness condition is shown to be as natural as is the notion of a Cauchy sequence in a metric space.

[The work on formal points is with Thierry Coquand, and follows papers by Josef Berger, Douglas Bridges, Hajime Ishihara, and Helmut Schwichtenberg; it further is related to recent insights of Maria Emilia Maietti and Giovanni Sambin.]

- SIMON HUBER, BASIL KARADAIS AND HELMUT SCHWICHTENBERG, *Towards a formal theory of computability*
Ludwig-Maximilians-Universität München

  We sketch a constructive formal theory TCFplus of computable functionals which allows to reason not only about the functionals themselves but also about their finite approximations. Types are built from base types by the formation of function types. The intended semantical domains for the base types are non-flat free algebras, given by their constructors, where the latter are injective and have disjoint ranges; both properties do not hold in the flat case. In this setting we give an informal proof (based on Berger (1993)) of Kreisel's density theorem (1959), and an adaption of Plotkin's definability theorem (1977, 1978)). It is shown that both proofs can be formalized in TCFplus.

- HIDEKI TSUIKI, *Domain Representations Derived from Dyadic Subbases*
Kyoto University

  Let $\mathbb{T}$ be the set $\{0, 1, \perp\}$, where $\perp$ is a special value which means undefinedness between 0 and 1. We consider the space $\mathbb{T}^\omega$ of infinite sequences of $\mathbb{T}$ and call each element of $\mathbb{T}^\omega$ a bottomed sequence. We call each appearance of 0 and 1 in a bottomed sequence a digit, and call a bottomed sequence with finite number of digits a finite bottomed sequence.

  A *dyadic subbase*[2] of a space X is a subbase $S = \{S_{n,i} : n < \omega, i < 2\}$ indexed with $\omega \times 2$ such that every element is a regular open set and $S_{n,1} = X \setminus \mathrm{cl}\ S_{n,0}$ for $n < \omega$. A dyadic subbase induces a

$\mathbb{T}^\omega$-coding of X, that is, the function $\varphi_S : X \to \mathbb{T}^\omega$ defined as

$$\varphi_S(x)(n) = \begin{cases} 0 & (x \in S_{n,0}) \\ 1 & (x \in S_{n,1}) \\ \bot & (otherwise) \end{cases}$$

is a topological embedding. Thus, a dyadic subbase induces a computational structure on X through a machine which input/output bottomed sequences.

In this talk, we show that approximation structures can also be derived from a dyadic subbase. We define

$$S(\sigma) = \bigcap_{n \in \mathrm{dom}(\sigma)} S_{n,\sigma(n)},$$
$$\bar{S}(\sigma) = \bigcap_{n \in \mathrm{dom}(\sigma)} \mathrm{cl}\ S_{n,\sigma(n)},$$

for a bottomed sequence σ. Here, $\mathrm{dom}(p) = \{n \in \omega : p(n) \neq \bot\}$. We say that a dyadic subbase is *proper* if cl $S(\sigma) = \bar{S}(\sigma)$ for every finite bottomed sequence σ.

From a dyadic subbase S, we construct an algebraic domain as follows. Let $K_S = \{\varphi_S(x)|_n : n < \omega, x \in X\}$ and D_S be its ideal completion. Here, $\sigma|_n$ is a finite bottomed sequence which is composed of the first n digits of σ. We show that if S is a proper dyadic subbase of a countably based metrizable space X, then φ_S embeds X in the minimal-limit set of D_S[1]. We also study uniformity structure [3] and bifinite structure of such a domain.

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- Wim Veldman, *The principle of open induction and some of its equivalents*
  Radboud University Nijmegen

A long list can be made of statements that are equivalent, in basic intuitionistic analysis, to Brouwer's Fan Theorem, see [1]. Brouwer's Fan Theorem, however, is only a rather weak consequence of his Bar Theorem. A presumably stronger consequence of the Bar Theorem is the following

> **Principle of Open Induction:** *Let A be an open subset of $[0,1]$. If, for every x in $[0,1]$, x belongs to A if every $y < x$ belongs to A, then A coincides with $[0,1]$.*

The Principle of Open Induction follows from the bar theorem and it implies the fan theorem. It turns out to be equivalent to the following contrapositive version of the

> **Bolzano-Weierstrass-theorem:** *Let $x_0, x_1, x_2, \ldots$ be an infinite sequence of real numbers such that, for every strictly increasing sequence γ of natural numbers there exists n such that $|x_{\gamma(n+1)} - x_{\gamma(n)}| > \frac{1}{2^n}$. Then, for each M in $\mathbb{R}$ there exists n such that $|x_n| > M$.*

It also turns out to be equivalent to the following principle that may be compared to the $\mathbf{\Sigma}^0_1$- *comprehension principle* in (classical) Reverse Mathematics:

> *Let B be an enumerable subset of $\mathbb{N}$. Suppose that, for every decidable subset A of $\mathbb{N}$, if $A \subseteq B$ and $\exists n[n \notin A]$, then $\exists n[n \notin A$ and $n \in B]$. Then $B = \mathbb{N}$.*

We intend to sketch proofs of these results and to discuss their significance for the project of Intuitionistic Reverse Mathematics.

References

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- Mariko Yasugi and Masako Washihara, *On sequential computability of a function*
  Kyoto Sangyo University

  We present a general treatment of the mutual relationship between two notions of "sequential computability" of a real function which is possibly Euclidean-discontinuous, one using limiting recursion and one using effective uniformity. Notwithstanding apparent differences of the two notions, they are equivalent under certain conditions, which "resemble" some properties holding in the Euclidean topology.

- Keita Yokoyama, *Comparing non-standard axioms with axioms of second-order arithmetic*
  Tohoku University

  It is known that we can use several non-standard techniques to do analysis within second-order arithmetic. On the other hand, H. J. Keisler introduced some non-standard axioms to characterize "big five systems" of second-order arithmetic. In this talk, I will introduce some more axioms for non-standard analysis such as "infinitesimal approximation principle", "transfer principle" or "saturation principle", and compare them with axioms of second-order arithmetic. Using these axioms, we can do reverse mathematics for non-standard analysis.