A Context Model for Constructing Membership Functions of Fuzzy Concepts Based on Modal Logic

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Abstract. In this paper we show that the context model proposed by Gebhardt and Kruse (1993) can be semantically extended and considered as a data model for constructing membership functions of fuzzy concepts within the framework of meta-theory developed by Resconi et al. in 1990s. Within this framework, we integrate context models by using a model of modal logic, and develop a method for calculating the expressions for the membership functions of composed fuzzy concepts based on values $\{0, 1\}$, which correspond to the truth values $\{F, T\}$ assigned to a given sentence as the response of a context considered as a possible world. It is of interest that fuzzy intersection and fuzzy union operators by this model are truth-functional, and, moreover, they form a well-known dual pair of *Product t-norm* T_P and *Probabilistic Sum t-conorm* S_P.

Keywords: Context model, fuzzy concept, membership function, modal logic

1 Introduction

The mathematical model of vague concepts was firstly introduced by Zadeh in 1965 by using the notion of membership functions resulted in the so-called theory of fuzzy sets. Since then mathematical foundations as well as successful applications of fuzzy set theory have already been developed (Klir & Yuan, 1995). As pointed out by Klir et al. (1997), these applications became feasible only when the methods of constructing membership functions of relevant fuzzy sets were efficiently developed in given application contexts.

In this paper we consider a context model, which was originally introduced by Gebhardt and Kruse (1993) in fuzzy data analysis, for constructing membership functions of vague concepts within framework of the modal logic based meta-theory developed by Resconi et al. (1992, 1993, 1996). By this approach, we can integrate context models by using a model of modal logic, and then develop a method for calculating the expressions for the membership functions of composed fuzzy concepts based on values $\{0, 1\}$ corresponding to the truth values $\{F, T\}$ assigned to a given sentence as the response of a context considered as a possible world. It is of interest to note that fuzzy intersection and fuzzy union operators by this model are truth-functional, and, moreover, they are a well-known dual pair of *Product t-norm* T_P and *Probabilistic Sum t-conorm* S_P.

The paper is organized as follows. In the next section, we briefly present some preliminary concepts: context model, modal logic, and meta-theory (with a short introduction to the modal logic interpretation of various uncertainty theories). In Section 3, we introduce a context model for fuzzy concept analysis and propose a model of modal logic for formulating fuzzy sets within a context model. Finally, some concluding remarks will be given in Section 4.

2 Preliminaries: Context model, Modal logic, and Meta-theory

2.1 Context Model

In the framework of fuzzy data analysis, Gebhardt and Kruse (1993) have introduced the context model as an approach to the representation, interpretation, and analysis of imperfect data. Shortly, the motivation of this approach stems from the observation that the origin of imperfect data is due to situations, where we are not able to specify an object by an original tuple of elementary characteristics because of the presence of incomplete statistical observations.

Formally, a context model is defined as a triple $\langle D, C, A_C(D)\rangle$, where D is a nonempty *universe of discourse*, C is a nonempty *finite set of contexts*, and the set $A_C(D) = \{a | a : C \to 2^D\}$ which is called the set of all vague characteristics of D with respect to C. For $a_1, a_2 \in A_C(D)$, then a_1 is said to be *more specific* than a_2 iff $(\forall c \in C)(a_1(c) \subseteq a_2(c))$.

If there is a finite measure P_C on the measurable space $(C, 2^C)$, then $a \in A_C(D)$ is called a *valuated vague characteristic* of D w.r.t. P_C. Then we call a quadruple $\langle D, C, A_C(D), P_C\rangle$ a valuated context model.

In this approach, each characteristic of an observed object is described by a fuzzy quantity formed by context model (Kruse et al. 1993). More refinements of the context model as well as its applications could be referred to Gebhardt and Kruse (1998), Gebhardt (2000). In the connection with formal concept analysis, it is interesting to note that in the case where C is a single-element set, say $C = \{c\}$, a context model formally becomes a formal context in the sense of Wille (see Ganter and Wille 1999) as follows. Let $\langle D, C, A_C(D)\rangle$ be a context model such that $|C| = 1$. Then the triple (O, A, R), where $O = D, A = A_C(D)$ and $R \subseteq O \times A$ such that $(o, a) \in R$ iff $o \in a(c)$, is a formal context. Thus, a context model can be considered as a collection of formal contexts. Huynh and Nakamori (2001) have considered and introduced the notion of fuzzy concepts within a context model and the membership functions associated with these fuzzy concepts. It is shown that fuzzy concepts can be interpreted exactly as the collections of α-cuts of their membership functions.

2.2 Modal logic

Propositional modal logic is an extension of classical propositional logic that adds to the propositional logic two unary modal operators, an operator of necessity, $\Box$, and an operator of possibility, $\Diamond$. Given a proposition p, $\Box p$ stands for the proposition "it is necessary that p", and similarly, $\Diamond p$ represents the proposition "it is possible that p". Modal logic is well developed syntactically (Chellas 1980).

In Resconi et al. (1992, 1993, 1996), the modal logic interpretation of various uncertainty theories is based on the fundamental semantics of modal logic using Kripke models.

A model, M, of modal logic is a triple $M = \langle W, R, V \rangle$, where W, R, V denote, respectively, a set of possible worlds, a binary relation on W, and a value assignment function, by which truth (T) or falsity (F) is assigned to each atom in each possible world, i.e.

$$V : W \times \mathcal{Q} \longrightarrow \{T, F\},$$

where $\mathcal{Q}$ is the set of all atoms. The value assignment function is inductively extended to all formulas in the usual way, the only interesting cases being

$$\begin{aligned} V(w, \Box p) = T &\Longleftrightarrow \forall w' \in W, (wRw') \Rightarrow V(w', p) = T \\ &\Longleftrightarrow \mathcal{R}_s(w) \subseteq \parallel p \parallel^M \end{aligned} \tag{1}$$

and

$$\begin{aligned} V(w, \Diamond p) = T &\Longleftrightarrow \exists w' \in W, (wRw') \text{ and } V(w', p) = T \\ &\Longleftrightarrow \mathcal{R}_s(w) \cap \parallel p \parallel^M \neq \emptyset \end{aligned} \tag{2}$$

where $\mathcal{R}_s(w) = \{w' \in W \mid wRw'\}$, and $\parallel p \parallel^M = \{w \mid V(w, p) = T\}$. The relation R is usually called an *accessibility relation*, and different systems of modal logic are characterised by different additional requirements on accessibility relation R. Some systems of modal logic are depicted as shown in Table 1.

Table 1. Acessibility relation and axiom schemas

No condition	**Df**$\Diamond$. $\Diamond p \leftrightarrow \neg\Box\neg p$
No condition	**K**. $\Box(p \to q) \to (\Box p \to \Box q)$
Serial: $\forall w \exists w'(wRw')$	**D**. $\Box p \to \Diamond p$
Reflexive: $\forall w(wRw)$	**T**. $\Box p \to p$
Symmetric: $\forall w \forall w'(wRw' \Rightarrow w'Rw)$	**B**. $p \to \Box\Diamond p$
Transitive: $\forall w \forall w' \forall w''(wRw'$ and $w'Rw'' \Rightarrow wRw'')$	**4**. $\Box p \to \Box\Box p$
Connected: $\forall w \forall w'(wRw'$ or $w'Rw)$	**4.3**. $\Box(\Diamond p \vee \Diamond q) \to (\Box\Diamond p \vee \Box\Diamond q)$
Euclidean: $\forall w \forall w' \forall w''(w'Rw$ and $w'Rw'' \Rightarrow wRw'')$	**5**. $\Diamond p \to \Box\Diamond p$

2.3 Meta-theory based upon modal logic

In a series of papers initiated by Resconi et al. (1992), the authors have developed a hierarchical uncertainty meta-theory based upon modal logic. Particularly, modal logic interpretations for several theories, including the mathematical theory of evidence[1], fuzzy set theory, possibility theory have been already proposed (Resconi et al. 1992, 1993, 1996; Harmanec et al. 1994, 1996; Klir and Harmanec 1994). These interpretations are based on Kripke models of modal logic. Moreover, Resconi et al. (1996) have suggested to add a weighting function $\Omega : W \to [0,1]$ such that $\sum_{i=1}^{n} \Omega(w_i) = 1$ as a component of model M. By such a way we obtain a new model $M_1 = \langle W, R, V, \Omega \rangle$.

With the model M_1, given a universe of discourse X we can consider propositions that are relevant to fuzzy sets having the following form

$$a_x : \text{“}x \text{ belongs to a given set } A\text{”}$$

where $x \in X$ and A denotes a subset of X that is based on a vague concept. Set A is then viewed as an ordinary fuzzy set whose membership function μ_A is defined, for all $x \in X$, by the following formula

$$\mu_A(x) = \sum_{i=1}^{n} \Omega(w_i)\, {}^i a_x$$

where

$${}^i a_x = \begin{cases} 1 & \text{if } V(w_i, a_x) = T, \\ 0 & \text{otherwise.} \end{cases}$$

The set-theoretic operations such as complement, intersection and union defined on fuzzy sets are then formulated within the model M_1 based on logical connectives NOT, AND, OR respectively (see Resconi et al. 1992, 1996).

To develop the interpretation of Dempster-Shafer theory of evidence (Shafer 1987) in terms of modal logic, Resconi et al. (1992) and Harmanec et al. (1994, 1996) employed propositions of the form

$$e_A : \text{“A given incompletely characterized element } \epsilon \text{ is classified in set } A\text{”}$$

where X denotes a frame of discernment, $A \in 2^X$ and $\epsilon \in X$. Due to the inner structure of these propositions, it is sufficient to consider as atomic propositions only propositions $e_{\{x\}}$, where $x \in X$. Furthermore, for each world $w_i \in W$, it is assumed that $V(w_i, e_{\{x\}}) = T$ for one and only one $x \in X$ and that the accessibility relation R is serial. Then the model M_1 yields the following equations for the basic functions in the Dempster-Shafer theory:

$$\begin{aligned} Bel(A) &= \textstyle\sum_{i=1}^{n} \Omega(w_i)\, {}^i(\Box e_A) \\ Pl(A) &= \textstyle\sum_{i=1}^{n} \Omega(w_i)\, {}^i(\Diamond e_A) \\ m(A) &= \textstyle\sum_{i=1}^{n} \Omega(w_i)\, {}^i[\Box e_A \wedge (\bigwedge_{x\in A} \Diamond e_{\{x\}})] \\ Com(A) &= \textstyle\sum_{i=1}^{n} \Omega(w_i)\, {}^i(\bigwedge_{x\in A} \Diamond e_{\{x\}}) \end{aligned}$$

[1] also called Dempster-Shafer theory

where Bel, Pl, m and Com denote the belief function, plausibility function, basic probability assignment, and commonality function in the Dempster-Shafer theory, respectively.

In the case where the basic probability assignment m in the Dempster-Shafer theory induces a nested family of focal elements, we obtain a special belief function called a *necessity measure*, along with a corresponding plausibility function called a *possibility measure* (Dubois and Prade 1987). It is shown by Klir and Harmanec (1994) that the accessibility relation R of models associated with possibility theory are transitive and connected, i.e. these models formally correspond to the modal system $S\mathbf{4.3}$ (see Table 1). The authors also showed the completeness of modal logic interpretation for possibility theory.

3 Fuzzy concepts by context model based on modal logic

3.1 Single domain case

As noted by Resconi and Turksen (2001), the specific meaning of a vague concept in a proposition may and usually does evaluate in different ways for different assessments of an entity by different agents, contexts, etc. For example, consider a sentence such as: "John is tall", where "tall" is a linguistic term of a linguistic variable, the height of people (Zadeh 1975). Assume that the domain $D = [0, 3m]$ which is associated with the base variable of the linguistic variable *height*. Note that in the terms of fuzzy sets, we may know John's height but must determine to what degree he is considered "tall". Next consider a set of worlds W in the sense of the Kripke model in which each world evaluates the sentence as either *true* or *false*. That is each world in W responds either as true or false when presented with the sentence "John is tall". These worlds may be contexts, agents, persons, etc. This implicitly shows that each world w_i in W determines a subset of D given as being compatible with the linguistic term *tall*. That is this subset represents w_i's view of the vague concept "tall". At this point we see that the context model introduced by Gebhardt and Kruse (1993) can be semantically extended and considered as a data model for constructing membership functions of vague concepts based on modal logic.

Let us consider a context model $\mathcal{C} = \langle D, C, A_C(D)\rangle$, where D is a domain of an attribute *at* which is applied to objects of concern, C is a non-empty finite set of contexts, and $A_C(D)$ is a set of linguistic terms associated with the domain D considered now as vague characteristics in the context model. For example, consider $D = [0, 3m]$ which is interpreted as the domain of the attribute *height* for people, C is a set of contexts such as Japanese, American, Swede, etc., and $A_C(D) = \{$ *very short, short, medium, tall, more or less tall,* $\ldots\}$. Each context determines a subset of D given as being compatible with a given linguistic term. Formally, each linguistic term can be considered as a mapping from C to 2^D. For linguistic terms such as *tall* and *very tall*, there are two interpretations possible: it may either be meant that *very tall* implies *tall*, i.e. that every very tall person is also tall. Or *tall* is an abbreviation for "tall, but not very tall". These two interpretations have been used in the literature depending on the shape of membership functions of relevant fuzzy sets. The linguistic term *very tall* is more specific than *tall* in the first interpretation, but not in the second one.

Furthermore, we can also associate with the context model a weighting function or a probability distribution Ω defined on C. As such we obtain a valuated context model $\mathcal{C} = \langle D, C, A_C(D), \Omega \rangle$.

By this context model, each linguistic term $a \in A_C(D)$ may be semantically represented by the fuzzy set A as follows

$$\mu_A(x) = \sum_{c \in C} \Omega(c)\mu_{a(c)}(x),$$

where $\mu_{a(c)}$ is the characteristic function of $a(c)$. Intuitively, while each subset $a(c)$, for $c \in C$, represents the c's view of the vague concept a, the fuzzy set A is the result of a weighted combinated view of the vague concept. For the sake of a further development in the next subsection, in the sequent we will formulate the problem in the terms of modal logic. To this end, we consider propositions that are relevant to a linguistic term have the following form

$$a_x : \text{“}x \text{ belongs to a given set } A\text{”},$$

where $x \in D$ and A denotes a subset of D that is based on a linguistic term a in $A_C(D)$. Assume that $C = \{c_1, \dots, c_n\}$, we now define a model of modal logic

$$M = \langle W, R, V_D, \Omega \rangle,$$

where $W = C$, that is each context c_i is associated with a possible world w_i; R is a binary relation on W, in this case R is the identity, i.e. each world w_i only itself is accessible; and V_D is the value assignment function such that for each world in W, by which truth (T) or falsity (F) is assigned to each atomic proposition a_x by

$$V_D(w_i, a_x) = \begin{cases} 1 & \text{if } x \in a(c_i), \\ 0 & \text{otherwise.} \end{cases}$$

We now define the compatible degree of any value x in the domain D to the linguistic term a (and the set A is then viewed as an ordinary fuzzy set) as the membership expression of truthood of the atomic sentence a_x in M as follows

$$\mu_A(x) = \sum_{i=1}^{n} \Omega(w_i)V_D(w_i, a_x) \tag{3}$$

Similar as in Resconi et al. (1996), it is straightforward to define the set-theoretic operations such as complement, intersection, union on fuzzy sets induced from linguistic terms in $A_C(D)$ by the model M using logical connectives NOT, AND, and OR respectively. Apply (3) to the complement A^c of fuzzy set A we have

$$\mu_{A^c}(x) = \sum_{i=1}^{n} \Omega(w_i)V_D(w_i, \neg a_x) = \sum_{i=1}^{n} \Omega(w_i)(1 - V_D(w_i, a_x)) = 1 - \mu_A(x).$$

In addition to propositions a_x, let us also consider propositions

$$b_x : \text{“}x \text{ belongs to a given set } B\text{”},$$

where $x \in D$ and B denotes a subset of D that is based on another linguistic term b in $A_C(D)$. To define composed fuzzy sets $A \cap B$ and $A \cup B$, we now apply logical connectives AND, OR to propositions a_x and b_x as follows

$$\mu_{A\cap B}(x) = \sum_{i=1}^{n} \Omega(w_i) V_D(w_i, a_x \wedge b_x) \tag{4}$$

$$\mu_{A\cup B}(x) = \sum_{i=1}^{n} \Omega(w_i) V_D(w_i, a_x \vee b_x) \tag{5}$$

It is easily seen that if a is more specific than b, we have

$$\mu_{A\cap B}(x) = \mu_A(x), \text{ and } \mu_{A\cup B}(x) = \mu_B(x),$$

this interpretation of linguistic hedges such as *very*, *less*, etc., is in accordance with that considered by Zadeh (1975).

Following properties of the operations $\vee, \wedge$ in classical logic, we easily obtain

$$\mu_{A\cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_{A\cap B}(x) \tag{6}$$

Furthermore, it follows directly by (4), (5) and (6) the following.

Proposition 3.1. *For any $x \in D$, we have*

$$\max(0, \mu_A(x) + \mu_B(x) - 1) \leq \mu_{A\cap B}(x) \leq \min(\mu_A(x), \mu_B(x))$$

$$\max(\mu_A(x), \mu_B(x)) \leq \mu_{A\cup B}(x) \leq \min(1, \mu_A(x) + \mu_B(x))$$

It should be noticed that under the constructive formulation of fuzzy sets by this context model, fuzzy intersection and fuzzy union operations are no longer truth-functional. Also, if there is a non-trivial relationship between contexts, we should take the relation R into account in defining of the fuzzy set A. A solution for this is by using modal operators $\Box$ and $\Diamond$, and results in an interval-valued fuzzy set defined as follows

$$\mu_A(x) = [\sum_{i=1}^{n} \Omega(w_i) V_D(w_i, \Box a_x), \sum_{i=1}^{n} \Omega(w_i) V_D(w_i, \Diamond a_x)].$$

In the next subsection we deal with the general case where composed fuzzy sets which represent linguistic combinations of linguistic terms of several context models are considered.

3.2 General case

It should be emphasized that Kruse et al. (1993) considered the same set of contexts for many domains of concern. While this assumption is acceptable in the framework of fuzzy data analysis where the characteristics (attributes) of observed objects are considered simultaneously in the same contexts, it may not be longer suitable for fuzzy concept analysis. For example, let us consider two attributes *Height* and *Income* of a set of people. Then, a set of contexts used for formulating of vague concepts of the attribute *Height* may be given as in the preceding subsection; while another set of contexts for formulating of vague concepts of the attribute *Income* (like *high*, *low*, etc.), may be given as a set of kinds of employees or a set of residential areas of employees.

Given two context models $\mathcal{C}_i = \langle D_i, C_i, A_{C_i}(D_i)\rangle$ defined on D_i, for $i = 1, 2$, respectively. A pair $(x, y) \in D_1 \times D_2$ is then interpreted as the pair of values of two attributes at_1 and at_2 for objects of concern. Recall that each element in $A_{C_i}(D_i)$ is a linguistic term understood as a mapping from $C_i \rightarrow 2^{D_i}$. Assume that $\mid C_i \mid= n_i$, for $i = 1, 2$.

We now define a unified Kripke model as follows: $M = \langle W, R, V, \Omega\rangle$, where $W = C_1 \times C_2$, R is the identity relation on W, and

$$\begin{aligned} \Omega : C_1 \times C_2 &\rightarrow [0, 1] \\ (c_i^1, c_j^2) &\mapsto \omega_{ij} = \omega_i \omega_j. \end{aligned}$$

where the simplified notations $\Omega(c_i^1, c_j^2) = \omega_{ij}, \Omega_1(c_i^1) = \omega_i, \Omega_2(c_j^2) = \omega_j$ are used.

For $a_i \in A_{C_i}(D_i)$, for $i = 1, 2$, we now formulate composed fuzzy sets, which represent combinated linguistic terms like *"a_1 and a_2"* and *"a_1 or a_2"* within model M.

For simplicity of notation, let us denote O a set of objects of concern which we may apply for two attributes at_1, at_2 those values range on domains D_1 and D_2, respectively. Then instead of considering fuzzy sets defined on different domains, we can consider fuzzy sets defined only on a universal set, the set of objects O. As such, we now consider atomic propositions of the form

$$a_o : \text{“An object } o \text{ is in relation to a linguistic term } a\text{”}$$

where $a \in A_{C_1}(D_1) \cup A_{C_2}(D_2)$ or a is a linguistic combination of linguistic terms in $A_{C_1}(D_1) \cup A_{C_2}(D_2)$.

Notice that this constructive formulation of composed fuzzy sets is comparable with the notion of the translation of a proposition a_o into a *relational assignment equation* introduced by Zadeh (1978).

Single term case. Firstly we consider the case where $a \in A_{C_1}(D_1)$. For this case, we define the valuation function V in M for atomic propositions a_o by

$$V((c_i^1, c_j^2), a_o) = \begin{cases} 1 & \text{if } at_1(o) \in a(c_i^1), \\ 0 & \text{otherwise,} \end{cases}$$

where $at_1(o) \in D_1$ denotes the value of attribute at_1 for object o.

Then the fuzzy set A which represents the meaning of the linguistic term a is defined in the model M as follows

$$\mu_A^M(o) = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \omega_{ij} V((c_i^1, c_j^2), a_o) \tag{7}$$

Set $W' = \{(c_i^1, c_j^2) \in C_1 \times C_2 \mid V((c_i^1, c_j^2), a_o) = 1\}$. It follows by definition of V that $W' = C_1' \times C_2$, where $C_1' = \{c_i^1 \in C_1 \mid at_1(o) \in a(c_i^1)\}$. Thus, we have

Proposition 3.2. *For any $o \in O$, we have $\mu_A^M(o) = \mu_A^{M_1}(o)$, where $\mu_A^{M_1}(o)$ is represented by $\mu_A^{M_1}(at_1(o))$ as in the preceding subsection.*

A similar result also holds for the case where $a \in A_{C_2}(D_2)$.

Composed term case. We now consider for the case where a is a composed linguistic term which is of the form like *"a_1 and a_2"* and *"a_1 or a_2"*, where $a_i \in A_{C_i}(D_i)$, for $i = 1, 2$. To formulate the composed fuzzy set A corresponding to the term a in the model M, we need to define the valuation function V for propositions a_o. It is natural to express a_o by

$$a_o = \begin{cases} a_{1,o} \vee a_{2,o} & \text{if } a \text{ is "} a_1 \text{ or } a_2 \text{"} \\ a_{1,o} \wedge a_{2,o} & \text{if } a \text{ is "} a_1 \text{ and } a_2 \text{".} \end{cases}$$

where $a_{i,o}$, for $i = 1, 2$, are propositions of the form

$a_{i,o}$: "An object o is in relation to a linguistic term a_i."

Consider the case where a is "a_1 *or* a_2". Then, the valuation function V for propositions a_o is defined as follows

$$V((c_i^1, c_j^2), a_{1,o} \vee a_{2,o}) = \begin{cases} 1 & \text{if } at_1(o) \in a_1(c_i^1) \vee at_2(o) \in a_2(c_j^2) \\ 0 & \text{otherwise.} \end{cases}$$

With this notation, we define the compatible degree of any object $o \in O$ to the composed linguistic term "a_1 *or* a_2" in the model M by

$$\mu_A(o) = \mu_{A_1 \cup A_2}(o) = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \omega_{ij} V((c_i^1, c_j^2), a_{1,o} \vee a_{2,o}) \tag{8}$$

where A_1, A_2 denote fuzzy sets which represent component linguistic terms a_1, a_2, respectively.

Similar for the case where a is "a_1 *and* a_2". The valuation function V for propositions a_o is then defined as follows

$$V((c_i^1, c_j^2), a_{1,o} \wedge a_{2,o}) = \begin{cases} 1 & \text{if } at_1(o) \in a_1(c_i^1) \wedge at_2(o) \in a_2(c_j^2) \\ 0 & \text{otherwise,} \end{cases}$$

and the compatible degree of any object $o \in O$ to the composed linguistic term "a_1 *and* a_2" in the model M is defined by

$$\mu_A(o) = \mu_{A_1 \cap A_2}(o) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \omega_{ij} V((c_i^1, c_j^2), a_{1,o} \wedge a_{2,o}) \tag{9}$$

Notice that in the case without the weighting function Ω in the model M, the membership expressions of composed fuzzy sets defined in (8) and (9) are comparable with those given by Resconi and Turksen (2001).

Now we examine the behaviours of operators $\cup, \cap$ in this formulation. Let us denote by

$$C_1' = \{c_i^1 \in C_1 \mid at_1(o) \in a_1(c_i^1)\},$$
$$C_2' = \{c_j^2 \in C_2 \mid at_2(o) \in a_2(c_j^2)\}.$$

It is easy to see that

$$V((c_i^1, c_j^2), (a_{1,o} \vee a_{2,o})) = \begin{cases} 1 & \text{if } (c_i^1, c_j^2) \in (C_1' \times C_2 \cup C_1 \times C_2'), \\ 0 & \text{otherwise}, \end{cases} \tag{10}$$

$$V((c_i^1, c_j^2), (a_{1,o} \wedge a_{2,o})) = \begin{cases} 1 & \text{if } (c_i^1, c_j^2) \in (C_1' \times C_2'), \\ 0 & \text{otherwise}. \end{cases} \tag{11}$$

Furthermore, we have the following representation

$$(C_1' \times C_2 \cup C_1 \times C_2') = (C_1' \times C_2 \uplus C_1 \times C_2') \setminus (C_1' \times C_2') \tag{12}$$

where $\uplus$ denotes an joint union which permits an iterative appearance of elements.

It is immediately to follow from (8)–(12) and Proposition 3.2 that

Proposition 3.3. *For any $o \in O$, we have*

$$\mu_{A_1 \cap A_2}(o) = \mu_{A_1}(o) \mu_{A_2}(o) \tag{13}$$

$$\mu_{A_1 \cup A_2}(o) = \mu_{A_1}(o) + \mu_{A_2}(o) - \mu_{A_1}(o) \mu_{A_2}(o) \tag{14}$$

Expressions (13) and (14) show that fuzzy intersection and fuzzy union operators by this model are truth-functional, and, moreover, they form a well-known dual pair of *Product t-norm T_P* and *Probabilistic Sum t-conorm S_P* (Klement 1997).

4 Conclusions

A context model for constructing membership functions of fuzzy concepts based on modal logic has been proposed in this paper. It has been shown that fuzzy intersection and fuzzy union operators by this model are truth-functional, and, more precisely, they form a well-known dual pair of Product t-norm T_P and Probabilistic Sum t-conorm S_P, respectively. It is worthwhile to note that for the purpose of finding new operators for using in the fuzzy expert system shell FLOPS, Buckley and Siler (1998) have used elementary statistical calculations on binary data for the truth of two fuzzy propositions to present new t-norm and t-conorm for computing the truth of AND, and OR propositions. Furthermore, their t-norm and t-conorm are also reduced to Product t-norm T_P and Probabilistic Sum t-conorm S_P in the case that the sample correlation coefficient equals to 0.

It should be worthwhile to note that the proposal in this paper can be developed as a method for evaluating queries, which contain vague predicates, in databases as well as for constructing membership functions for fuzzy concepts in mining fuzzy association rules from databases (Hong et al. 1999; Kuok et al. 1998). These problems are being the subject of our further work.

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