Multiple Attribute Decision Making Under Uncertainty: The Evidential Reasoning Approach Revisited

Van Nam Huynh, Yoshiteru Nakamori, *Member, IEEE*, Tu Bao Ho, *Member, IEEE*, Tetsuya Murai

***Abstract*— In multiple-attribute decision making (MADM) problems, one often needs to deal with decision information with uncertainty. During the last decade, Yang and Singh (1994) have proposed and developed an evidential reasoning (ER) approach to deal with such MADM problems. Essentially, this approach is based on an evaluation analysis model and the Dempster's rule of combination in the Dempster-Shafer theory of evidence.**

In this paper, we re-analyse the ER approach explicitly in terms of Dempster-Shafer theory, and then propose a general scheme of attribute aggregation in MADM under uncertainty. In the spirit of such a reanalysis, the previous ER algorithms are reviewed and other two aggregation schemes are discussed. Concerning the synthesis axioms recently proposed by Yang and Xu (2002) for which a rational aggregation process should grant, theoretical features of new aggregation schemes are also explored thoroughly. A numerical example traditionally examined in published sources on the ER approach is used to illustrate the discussed techniques.

I. Introduction

Practically, decision makers are often required to choose between several alternatives or options where each option exhibits a range of attributes of both a quantitave and qualitative nature. A decision may not be properly made without fully taking into account all attributes concerned [3], [9], [16], [22], [28], [33]. In addition, in many MADM problems, one also frequently needs to deal with decision knowledge represented in forms of both qualitative and quantitative information with uncertainty.

So far, many attempts have been made to integrate techniques from artificial intelligence (AI) and operational research (OR) for handling uncertain information, e.g., [1], [4], [5], [6], [10], [11], [15], [24], [27]. During the last decade or so, an evidential reasoning (ER) approach has been proposed and developed for MADA under uncertainty in [28], [29], [31], [32], [33]. Essentially, this approach is based on an evaluation analysis model [35] and the evidence combination rule of the Dempster-Shafer (D-S) theory [20] (which in turn is one of the major techniques for dealing with uncertainty in AI). The ER approach has been applied to a range of MADM problems in engineering and management, including motorcycle assessment [29], general cargo ship design [18], system safety analysis and synthesis [23], retro-fit ferry design [30] among others.

V.N. Huynh, Y. Nakamori and T.B. Ho are with the Japan Advanced Institute of Science and Technology, Tatsunokuchi, Ishikawa 923-1292, Japan. Email: {huynh, nakamori, bao}@jaist.ac.jp.

T. Murai is with Graduate School of Engineering, Hokkaido University, Kita 13, Nishi 8, Kita-ku, Sapporo 060-8628, Japan

The kernel of the ER approach is an ER algorithm developed on the basis of a multi-attribute evaluation framework and Dempster's rule of combination in D-S theory of evidence [28]. Basically, the algorithm makes use of Dempster's rule of combination to aggregate attributes of a multi-level structure. Due to a need of developing theoretically sound methods and tools for dealing with MADM problems under uncertainty, recently, Yang and Xu [33] have proposed a system of four synthesis axioms within the ER assessment framework with which a rational aggregation process needs to satisfy. It has also been shown that the original ER algorithm only satisfies these axioms approximately. At the same time, guided by the aim exactly, the authors have proposed a new ER algorithm that satisfies all the synthesis axioms precisely.

Interestingly enough, the D-S theory of evidence on the one hand allows us to coarse or refine the data by changing to a higher or lower level of granularity (or attribute in the context of a multi-level structure) accompanied with a powerful evidence combination rule. This is an essential feature for multiple attribute assessment systems based on a multi-level structure of attributes. On the other hand, one of major advantages of the D-S theory over conventional probability is that it provides a straightforward way of quantifying ignorance and is therefore a suitable framework for handling incomplete uncertain information. This is especially important and useful for dealing with uncertain subjective judgments when multiple basic attributes (also called factors) need to be considered simultaneously [28].

It is worth emphasizing that the underlying basis of using Dempster's rule of combination is the independent assumption of information sources to be combined. However, in situations of multiple attribute assessment based on a multi-level structure of attributes, assumptions regarding the independence of attributes' uncertain evaluations may not be appropriate in general. Moreover, another important issue concerning the rule is that it may yield counterintuitive results especially when a high conflict between information sources to be combined arises. This problem of completely ignoring conflict caused by a normalization in Dempster's rule was originally pointed out in [34]. Consequently, this has highly motivated researchers to propose a number of other combination rules in the literature to address the problem, e.g. [24], [26] (see [19] for a recent survey).

In this paper we deal with the attribute aggregation problem in the ER approach to MADM under uncertainty developed in, e.g., [28], [33]. First we reanalysis the previous ER approach in terms of D-S theory so that the attribute aggregation problem in MADM under uncertainty can be generally formulated as a problem of evidence combination. Then we propose several new aggregation schemes and simultaneously examine their theoretical features. For the purpose of the present paper, we take only qualitative attributes of an MADM problem with uncertainty into account, though quantitave attributes would be also included in a similar way as considered in [28], [29].

To proceed, it is first necessary to briefly recall basic notions on the MADM problem with uncertainty, the basic evaluation framework and the D-S theory of evidence. This is undertaken in Section II and followed in Section III by a discussion of the ER approach to MADM under uncertainty proposed previously. Section IV then explores the attribute aggregation problem detailedly, and Section V examines a motorcycle performance assessment problem taken from [33]. Finally, Section IV presents some concluding remarks.

II. Background

A. Problem Description

This subsection describes an MADM problem with uncertainty through a tutorial example taken from [33]. As mentioned above, for the purpose of this paper, only qualitative attributes of the problem are taken into account. For more details the reader could be referred to [28], [29].

To subjectively evaluate qualitative attributes (or features) of alternatives (or options), a set of evaluation grades may be firstly supplied as follows

$$\mathcal{H} = \{H_1, \ldots, H_n, \ldots, H_N\}$$

where H_n's are called evaluation grades to which the state of a qualitative attribute y may be evaluated. That is, $\mathcal{H}$ provides a complete set of distinct standards for assessing qualitative attributes in question. Although different attributes may have different sets of evaluation grades, for the sake of simplicity, in this paper we assume the same set $\mathcal{H}$ for all attributes of concern. Further, without loss of generality, it is assumed that H_{n+1} is preferred to H_n.

Let us turn to a problem of motorcycle evaluation [7]. To evaluate the quality of the *operation* of a motorcycle, the set of distinct evaluation grades is defined by (1) at the top of the page.

Because *operation* is a general technical concept and is not easy to evaluate directly, it needs to be decomposed into detailed concepts such as *handling*, *transmission*, and *brakes*. Again, if a detailed concept is still too general to assess directly, it may be further decomposed into more detailed concepts. For example, the concept of *brakes* is measured by *stopping power*, *braking stability*, and *feel at control*, which can probably be directly evaluated by an expert and therefore referred to as basic attributes (or basic factors).

Generally, a qualitative attribute y may be evaluated through a hierarchical structure of its subattributes. For instance, the hierarchy for evaluation of the *operation* of a motorcycle is depicted as in Fig. 1.

In evaluation of qualitative attributes, judgments could be uncertain. For example, in the problem of evaluating different types of motorcycles, the following type of uncertain subjective judgments for the *brakes* of a motorcycle, say "Yamaha", was frequently used [7], [33]:

1) Its *stopping power* is *average* with a confidence degree of 0.3 and it is *good* with a confidence degree of 0.6.
2) Its *braking stability* is *good* with a confidence degree of 1.
3) Its *feel at control* is evaluated to be *good* with a confidence degree of 0.5 and to be *excellent* with a confidence degree of 0.5.

In the above statements, the confidence degrees represent the uncertainty in the evaluation. Note that the total confidence degree in each statement may be smaller than 1 as the case of the first statement. This may be due to incomplete of available information.

In a similar fashion, all basic attributes in question could be evaluated. The problem now is how to generate an overall assessment of the *operation* of a motorcycle by aggregating the all uncertain judgments of its basic attributes in a rational way. The evidential reasoning approach developed in [28], [29], [33] has provided a means based on Dempster's rule of combination for dealing with such an aggregation problem.

B. Evaluation Analysis Model

The evaluation analysis model was proposed in [35] to represent uncertain subjective judgments, such as statements specified in preceding subsection, in a hierarchical structure of attributes.

To begin with, let us suppose a simple hierarchical structure consisting of two levels with a general attribute, denoted by y, at the top level and a finite set E of its basic attributes at the bottom level (graphically, shown in Fig. 2). Let

$$E = \{e_1, \ldots, e_i, \ldots, e_L\}$$

and assume the weights of basic attributes are given by $W = (w_1, \ldots, w_i, \ldots, w_L)$, where w_i is the relative weight of the ith basic attribute (e_i) with $0 \leq w_i \leq 1$. Attribute weights essentially play an important role in multi-attribute decision models. Because the elicitation of weights can be difficult, several methods have been proposed for reducing the burden of the process [14].

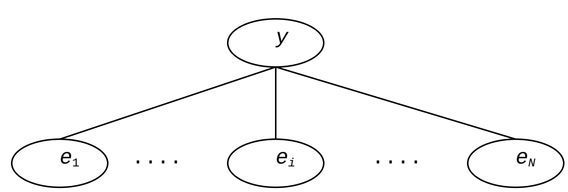


Fig. 2. A two-level hierarchy

Given the following set of evaluation grades

$$\mathcal{H} = \{H_1, \ldots, H_n, \ldots, H_N\}$$

designed as distinct standards for assessing an attribute, then an assessment for e_i of an alternative can be mathematically represented in terms of the following distribution [33]

$$S(e_i) = \{(H_n, \beta_{n,i}) \mid n = 1, \ldots, N\}, \text{ for } i = 1, \ldots, L \quad (2)$$

$$\mathcal{H} = \{\textit{poor}\ (H_1), \textit{indifferent}\ (H_2), \textit{average}\ (H_3), \textit{good}\ (H_4), \textit{excellent}\ (H_5)\} \quad (1)$$

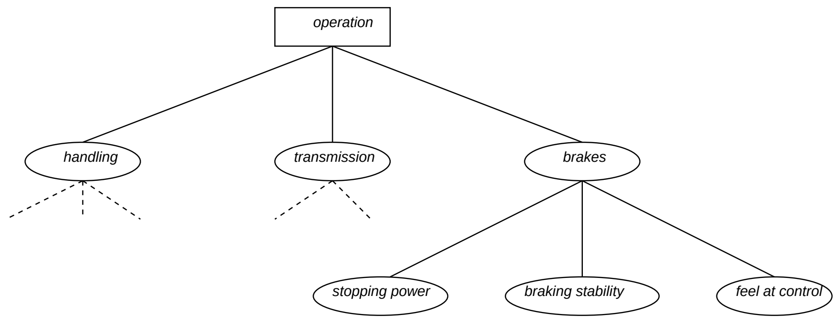


Fig. 1. Evaluation hierarchy for *operation* [33]

where $\beta_{n,i}$ denotes a degree of belief satisfying $\beta_{n,i} \geq 0$, and $\sum_{n=1}^{N} \beta_{n,i} \leq 1$. An assessment $S(e_i)$ is called *complete* (respectively, *incomplete*) if $\sum_{n=1}^{N} \beta_{n,i} = 1$ (respectively, $\sum_{n=1}^{N} \beta_{n,i} < 1$).

For example, the three assessments 1.–3. given in preceding subsection can be represented in the form of distributions defined by (2) as

$$\begin{aligned} S(\textit{stopping power}) &= \{(H_3, 0.3), (H_4, 0.6)\} \\ S(\textit{braking stability}) &= \{(H_4, 1)\} \\ S(\textit{feel at control}) &= \{(H_4, 0.5), (H_5, 0.5)\} \end{aligned}$$

where only grades with nonzero degrees of belief are listed in the distributions.

Let us denote β_n the degree of belief to which the general attribute y is assessed to the evaluation grade of H_n. The problem now is to generate β_n, for $n = 1, \ldots, N$, by combinating the assessments for all associated basic attributes e_i $(i = 1, \ldots, L)$ as given in (2). However, before continuing the discussion, it is necessary to briefly review the basis of D-S theory of evidence in the next subsection.

C. Dempster-Shafer Theory of Evidence

In D-S theory, a problem domain is represented by a finite set Θ of mutually exclusive and exhaustive hypotheses, called *frame of discernment* [20]. In the standard probability framework, all elements in Θ are assigned a probability. And when the degree of support for an event is known, the remainder of the support is automatically assigned to the negation of the event. On the other hand, in D-S theory mass assignments are carried out for events as they know, and committing support for an event does not necessarily imply that the remaining support is committed to its negation. Formally, a basic probability assignment (BPA, for short) is a function $m : 2^{\Theta} \to [0, 1]$ verifying

$$m(\emptyset) = 0, \text{ and } \sum_{A \in 2^{\Theta}} m(A) = 1$$

The quantity $m(A)$ can be interpreted as a measure of the belief that is committed exactly to A, given the available evidence. A subset $A \in 2^{\Theta}$ with $m(A) > 0$ is called a *focal element* of m. A BPA m is called to be *vacuous* if $m(\Theta) = 1$ and $m(A) = 0$ for all $A \neq \Theta$.

Two evidential functions derived from the basic probability assignment are the belief function Bel and the plausibility function Pl, defined as

$$Bel(A) = \sum_{\emptyset \neq B \subseteq A} m(B), \text{ and } Pl(A) = \sum_{B \cap A \neq \emptyset} m(B)$$

The difference between $m(A)$ and $Bel(A)$ is that while $m(A)$ is our belief committed to the subset A excluding any of its proper subsets, $Bel(A)$ is our degree of belief in A as well as all of its subsets. Consequently, $Pl(A)$ represents the degree to which the evidence fails to refute A. Note that all the three functions are in an one-to-one correspondence with each other.

Two useful operations that play a central role in the manipulation of belief functions are *discounting* and *Dempster's rule of combination* [20]. The discounting operation is used when a source of information provides a BPA m, but one knows that this source has probability α of reliable. Then one may adopt $(1 - \alpha)$ as one's *discount rate*, which results in a new BPA m^{α} defined by

$$m^{\alpha}(A) = \alpha m(A), \quad \text{for any } A \subset \Theta \quad (3)$$
$$m^{\alpha}(\Theta) = (1 - \alpha) + \alpha m(\Theta) \quad (4)$$

Consider now two pieces of evidence on the same frame Θ represented by two BPAs m_1 and m_2. Dempster's rule of combination is then used to generate a new BPA, denoted by $(m_1 \oplus m_2)$ (also called the orthogonal sum of m_1 and m_2), defined as follows

$$\begin{aligned} (m_1 \oplus m_2)(\emptyset) &= 0, \\ (m_1 \oplus m_2)(A) &= \frac{1}{1-\kappa} \sum_{B \cap C = A} m_1(B) m_2(C) \end{aligned} \quad (5)$$

where

$$\kappa = \sum_{B \cap C = \emptyset} m_1(B) m_2(C) \quad (6)$$

Note that the orthogonal sum combination is only applicable to such two BPAs that verify the condition $\kappa < 1$.

As we will partially see in the following sections, these two operation essentially play an important role in the ER approach to MADM under uncertainty developed in, e.g., [28], [29], [33]. Although the discounting operation has not been mentioned explicitly in these published sources.

III. The Evidential Reasoning Approach

Let us return to the two-level hierarchical structure with a general attribute y at the top level and a finite set $E = \{e_1, \ldots, e_i, \ldots, e_L\}$ of its basic attributes at the bottom level. Let us be given weights w_i ($i = 1, \ldots, L$) of basic attributes e_i ($i = 1, \ldots, L$), respectively. Denote β_n the degree of belief to which the general attribute y is assessed to the evaluation grade of H_n, for $n = 1, \ldots, N$.

A. The Original ER Algorithm

The original ER algorithm proposed in [28] has been used for the purpose of obtaining β_n ($n = 1, \ldots, N$) by aggregating the assessments of basic attributes given in (2). The summary of the algorithm in this subsection is taken from [33].

Given the assessment $S(e_i)$ of a basic attribute e_i ($i = 1, \ldots, L$), let $m_{n,i}$ be a basic probability mass representing the belief degree to which the basic attribute e_i supports the hypothesis that the attritute y is assessed to the evaluation grade H_n. Let $m_{\mathcal{H},i}$ be the remaining probability mass unassigned to any individual grade after all the N grades have been considered for assessing the general attribute y as far as e_i is concerned. These quantities are defined as follows

$$m_{n,i} = w_i \beta_{n,i}, \text{ for } n = 1, \ldots, N \tag{7}$$

$$m_{\mathcal{H},i} = 1 - \sum_{n=1}^{N} m_{n,i} = 1 - w_i \sum_{n=1}^{N} \beta_{n,i} \tag{8}$$

Let $E_{I(i)} = \{e_1, \ldots, e_i\}$ be the subset of first i basic attributes. Let $m_{n,I(i)}$ be a probability mass defined as the belief degree to which all the basic attributes in $E_{I(i)}$ supports the hypothesis that y is assessed to H_n. Let $m_{\mathcal{H},I(i)}$ be the remaining probability mass unassigned to individual grades after all the basic attributes in $E_{I(i)}$ have been assessed. The quantities $m_{n,I(i)}$ and $m_{\mathcal{H},I(i)}$ can be generated by combining the basic probability masses $m_{n,j}$ and $m_{\mathcal{H},j}$ for all $n = 1, \ldots, N$, and $j = 1, \ldots, i$.

With these notations, the key step in the original ER algorithm is to inductively calculate $m_{n,I(i+1)}$ and $m_{\mathcal{H},I(i+1)}$ as follows

$$\begin{aligned} m_{n,I(i+1)} &= K_{I(i+1)}(m_{n,I(i)} m_{n,i+1} + m_{n,I(i)} m_{\mathcal{H},i+1} \\ &\quad + m_{\mathcal{H},I(i)} m_{n,i+1}) \end{aligned} \tag{9}$$

$$m_{\mathcal{H},I(i+1)} = K_{I(i+1)}(m_{\mathcal{H},I(i)} m_{\mathcal{H},i+1}) \tag{10}$$

for $n = 1, \ldots, N$, $i = 1, \ldots, L-1$, and $K_{I(i+1)}$ is a normalizing factor defined by

$$K_{I(i+1)} = \left[1 - \sum_{t=1}^{N} \sum_{\substack{j=1 \\ j \neq t}}^{N} m_{t,I(i)} m_{j,i+1} \right]^{-1} \tag{11}$$

Then we obtain

$$\begin{aligned} \beta_n &= m_{n,I(L)}, \text{ for } n = 1, \ldots, N \\ \beta_{\mathcal{H}} &= m_{\mathcal{H},I(L)} = 1 - \sum_{n=1}^{N} \beta_n \end{aligned} \tag{12}$$

B. Synthesis Axioms and the Modified ER Algorithm

Inclined to developing theoretically sound methods and tools for dealing with MADM problems under uncertainty, Yang and Xu [33] have recently proposed a system of four synthesis axioms within the ER assessment framework with which a rational aggregation process needs to satisfy. These axioms are symbolically stated as below.

Axiom 1. (*Independency*) If $\beta_{n,i} = 0$ for all $i = 1, \ldots, L$, then $\beta_n = 0$.

Axiom 2. (*Consensus*) If $\beta_{k,i} = 1$ and $\beta_{n,i} = 0$, for all $i = 1, \ldots, L$, and $n = 1, \ldots, N$, $n \neq k$, then $\beta_k = 1$, $\beta_n = 0$, for $n = 1, \ldots, N$, $n \neq k$.

Axiom 3. (*Completeness*) Assume $\mathcal{H}^+ \subset \mathcal{H}$ and denote $J^+ = \{n | H_n \in \mathcal{H}^+\}$. If $\beta_{n,i} > 0$ for $n \in J^+$ and $\sum_{n \in J^+} \beta_{n,i} = 1$, for all $i = 1, \ldots, L$, then $\beta_n > 0$ for $n \in J^+$ and $\sum_{n \in J^+} \beta_n = 1$ as well.

Axiom 4. (*Incompleteness*) If there exists $i \in \{1, \ldots, L\}$ such that $\sum_{n=1}^{N} \beta_{n,i} < 1$, then $\sum_{n=1}^{N} \beta_n < 1$.

It is easily seen from (9–12) that the original ER algorithm naturally follows the independency axiom. Concerning the second axiom, the following theorem is due to Yang and Xu [33].

Theorem 1: If β_n and $\beta_{\mathcal{H}}$ are calculated using (12), then the concensus axiom holds if and only if

$$\prod_{i=1}^{L} (1 - w_i) = 0 \tag{13}$$

Note that the only constraint imposed on the weights w_i ($i = 1, \ldots, L$) in the ER approach is $0 \leq w_i \leq 1$. By Theorem 1, it implies that if $w_i = 1$ then e_i dominates the assessment of y, i.e. other basic attributes with smaller weights play no role in the assessment. To resolve this dilemma, the following scheme for weight normalization has been considered in [28], [29], [30]

$$\overline{w}_i = \alpha \frac{w_i}{\max\limits_{i=1,\ldots,L} \{w_i\}} \tag{14}$$

and α is a constant determined by satisfying

$$\prod_{i=1}^{L} \left(1 - \alpha \frac{w_i}{\max\limits_{i=1,\ldots,L} \{w_i\}} \right) \leq \delta$$

where δ is a small constant representing the degree of approximation in aggregation. By considering normalized weights $\overline{w}_i$'s instead of w_i's, it means the consensus axiom could only be satisfied approximately. However, this weight normalization has still another shortcoming that the most important attribute may play a dominating role in the assessment of y. Further, it has been also shown in [33] that the original ER algorithm does not satisfy the completeness axiom.

Under such a consideration, Yang and Xu [33] proposed a new ER algorithm that satisfies all the synthesis axioms. Its main features are summarized as follows

1) *Weight normalization*. In the new ER algorithm, the weights w_i ($i = 1, \ldots, L$) of basic attributes are nor-

malized such that: $0 \leq w_i \leq 1$, and

$$\sum_{i=1}^{L} w_i = 1 \tag{15}$$

2) *Aggregation process*. First, the probability mass $m_{\mathcal{H},i}$ given in (8) is decomposed into two parts: $m_{\mathcal{H},i} = \tilde{m}_{\mathcal{H},i} + \overline{m}_{\mathcal{H},i}$, where

$$\overline{m}_{\mathcal{H},i} = 1 - w_i, \text{ and } \tilde{m}_{\mathcal{H},i} = w_i \left(1 - \sum_{n=1}^{N} \beta_{n,i}\right) \tag{16}$$

Then, with the notations as in preceding section, the process of aggregating the first i assessments with the $(i+1)$th assessment is recursively carried out as follows

$$\begin{aligned} m_{n,I(i+1)} &= K_{I(i+1)}[m_{n,I(i)}m_{n,i+1} + m_{n,I(i)}m_{\mathcal{H},i+1} \\ &\quad + m_{\mathcal{H},I(i)}m_{n,i+1}] && (17)\\ m_{\mathcal{H},I(i)} &= \tilde{m}_{\mathcal{H},I(i)} + \overline{m}_{\mathcal{H},I(i)}, \ n = 1,\ldots,N \\ \tilde{m}_{\mathcal{H},I(i+1)} &= K_{I(i+1)}[\tilde{m}_{\mathcal{H},I(i)}\tilde{m}_{\mathcal{H},i+1} \\ &\quad + \overline{m}_{\mathcal{H},I(i)}\tilde{m}_{\mathcal{H},i+1} + \tilde{m}_{\mathcal{H},I(i)}\overline{m}_{\mathcal{H},i+1}] && (18)\\ \overline{m}_{\mathcal{H},I(i+1)} &= K_{I(i+1)}[\overline{m}_{\mathcal{H},I(i)} + \overline{m}_{\mathcal{H},i+1}] && (19) \end{aligned}$$

where $K_{I(i+1)}$ is defined as same as in (11).
For assigning the assessment $S(y)$ for the general attribute y, after all L assessments of basic attributes have been aggregated, the algorithm finally defines

$$\begin{aligned} \beta_n &= \frac{m_{n,I(L)}}{1 - \overline{m}_{\mathcal{H},I(L)}}, \text{ for } n = 1,\ldots,N && (20)\\ \beta_{\mathcal{H}} &= \frac{\tilde{m}_{\mathcal{H},I(L)}}{1 - \overline{m}_{\mathcal{H},I(L)}} && (21) \end{aligned}$$

and then,

$$S(y) = \{(H_n, \beta_n), n = 1,\ldots,N\} \tag{22}$$

The following theorems are due to Yang and Xu [33] that are taken for granted to develop the new ER algorithm above.

Theorem 2: The degrees of belief defined by (20) and (21) satisfy the following

$$\begin{aligned} &0 \leq \beta_n, \beta_{\mathcal{H}} \leq 1, \quad n = 1,\ldots,N \\ &\sum_{n=1}^{N} \beta_n + \beta_{\mathcal{H}} = 1 \end{aligned}$$

Theorem 3: If $\beta_{k,i} = 1$ and $\beta_{n,i} = 0$ for all $n = 1,\ldots,N$ with $n \neq k$, and $i = 1,\ldots,L$, then $\beta_k = 1$, $\beta_n = 0$ for all $n = 1,\ldots,N$ with $n \neq k$, and $\beta_{\mathcal{H}} = 0$.

Theorem 4: Let $\mathcal{H}^+ \subset \mathcal{H}$ and $\mathcal{H}^- = \mathcal{H} \setminus \mathcal{H}^+$, we denote $J^+ = \{n | H_n \in \mathcal{H}^+\}$ and $J^- = \{j | H_j \in \mathcal{H}^-\}$. If $\beta_{n,i} > 0$ ($n \in J^+$), $\sum_{n\in J^+} \beta_{n,i} = 1$ and $\beta_{j,i} = 0$ ($j \in J^-$), for all $i = 1,\ldots,L$, then $\sum_{n\in J^+} \beta_n = 1$ and $\beta_j = 0$ ($j \in J^-$).

Theorem 5: Assume that $0 < w_i < 1$ for all $i = 1,\ldots,L$. If there exists an i such that $\sum_{n=1}^{N} \beta_{n,i} < 1$, then $\beta_{\mathcal{H}} > 0$.

In [33], the authors have given direct proofs of these theorems, which are somehow complicated. In the next section, we also give a brief description of these theorems in terms of D-S theory.

IV. A Reanalysis of the ER Approach

Let us remind ourselves the available information given to an assessment problem in the two-level hierarchical structure as depicted in Fig. 2:

- the assessments $S(e_i)$ for basic attributes e_i ($i = 1,\ldots,L$), and
- the weights w_i of the basic attributes e_i ($i = 1,\ldots,L$).

Given the assessment $S(e_i)$ of a basic attribute e_i ($i = 1,\ldots,L$), we now define a corresponding BPA, denoted by m_i, which quantifies the belief about the performance of e_i as follows: for any $H \subseteq \mathcal{H}$

$$m_i(H) = \begin{cases} \beta_{n,i}, & \text{if } H = \{H_n\} \\ (1 - \sum_{n=1}^{N} \beta_{n,i}), & \text{if } H = \mathcal{H} \\ 0, & \text{otherwise} \end{cases} \tag{23}$$

For the sake of simplicity, we will write $m_i(H_n)$ instead of $m_i(\{H_n\})$ as usual. The quantity $m_i(H_n)$ represents the belief degree that supports for the hypothesis that e_i is assessed to the evaluation grade H_n. While $m_i(\mathcal{H})$ is the remaining probability mass unassigned to any individual grade after all evaluation grades have been considered for assessing e_i. If $S(e_i)$ is a complete assessment, m_i is a probability distribution, i.e. $m_i(\mathcal{H}) = 0$. Otherwise, $m_i(\mathcal{H})$ quantifies the ignorance.

As such with L basic attributes e_i, we obtain L corresponding BPAs m_i as quantified beliefs of the assessments for basic attributes. The problem now is how to generate an assessment for y, i.e. $S(y)$, represented by a BPA m, from m_i and w_i ($i = 1,\ldots,L$). Formally, we aim at obtaining a BPA m that combines all m_i's with taking weights w_i's into account in the general form of the following

$$m = \bigoplus_{i=1}^{L} (w_i \otimes m_i) \tag{24}$$

where $\otimes$ is a product-type operation and $\oplus$ is a sum-type operation in general.

As such, by applying different particular operations for $\otimes$ and $\oplus$, we may have different aggregation schemes for obtaining the BPA m represented the generated assessment $S(y)$. However, before exploring any new aggregation schemes, we first interestingly re-interpret the original ER approach in the spirit of the new formulation.

A. The Discounting-and-Orthogonal Sum Scheme

Let us first consider $\otimes$ as the discounting operation and $\oplus$ as the orthogonal sum in D-S theory. Then, for each $i = 1,\ldots,L$, we have $(w_i \otimes m_i)$ is a BPA (refer to (3–4)) defined by (25), for any $H \subseteq \mathcal{H}$ and $n = 1,\ldots,N$, at the top of the next page.

With this formulation, we consider each m_i as the belief quatified from the information source $S(e_i)$ and the weight w_i as a "degree of trust" of $S(e_i)$ supporting the assessment of y as a whole. As mentioned in [20], an obvious way to use discounting with Dempster's rule of combination is to discount all BPAs m_i ($i = 1,\ldots,L$) at corresponding rates $(1 - w_i)$ ($i = 1,\ldots,L$) before combining them.

$$(w_i \otimes m_i)(H) \triangleq m_i^{w_i}(H) = \begin{cases} w_i \beta_{n,i}, & \text{if } H = \{H_n\} \\ (1 - w_i) + w_i(1 - \sum_{n=1}^{N} \beta_{n,i}) = 1 - w_i \sum_{n=1}^{N} \beta_{n,i}, & \text{if } H = \mathcal{H} \\ 0, & \text{otherwise} \end{cases} \tag{25}$$

Thus, Dempster's rule of combination now allows us to combine BPAs $m_i^{w_i}$ $(i = 1, \ldots, L)$ under the independent assumption of information sources for generating the BPA m for the assessment of y. Namely,

$$m = \bigoplus_{i=1}^{L} m_i^{w_i} \tag{26}$$

where, with an abuse of the notation, $\oplus$ stands for the orthogonal sum.

At this juncture, we can see that the aggregation processes in original ER approach reviewed above follow this discounting-and-orthogonal sum scheme. In addition, it is of interest to note that, by definition, in this aggregation scheme it would be not necessary to require the process of weight normalization satisfying the constraint

$$\sum_{i=1}^{L} w_i = 1$$

That is, by relaxing this constraint on weights, we may avoid the mutual and exclusive assumption of information sources supporting the assessment for y, which seems to be questionable in the context of aggregated assessments, even though weights would have a probability-like interpretation.

It would be worth noting that two BPAs $m_i^{w_i}$ and $m_j^{w_j}$ are combinable, i.e. $(m_i^{w_i} \oplus m_j^{w_j})$ does exist, if and only if

$$\kappa = \sum_{t=1}^{N} \sum_{\substack{n=1 \\ n \neq t}}^{N} m_i^{w_i}(H_n) m_j^{w_j}(H_t) < 1$$

For example, assume that we have two basic attributes e_1 and e_2 with

$$\begin{aligned} S(e_1) &= \{(H_1, 0), (H_2, 0), (H_3, 0), (H_4, 1), (H_5, 0)\} \\ S(e_2) &= \{(H_1, 0), (H_2, 0), (H_3, 1), (H_4, 0), (H_5, 0)\} \end{aligned}$$

and both are equally important, or $w_1 = w_2$. In the extreme, setting $\delta = 0$ and the weights w_1 and w_2 being normalized using (14) lead to $w_1 = w_2 = 1$, which results in $(m_1^{w_1} \oplus m_2^{w_2})$ does not exist.

Another issue with the orthogonal sum operation is in using the total probability mass κ (called the degree of conflict [24]) associated with conflict as defined in the normalization factor. Consequently, applying it in an aggregation process may yield counterintuitive results in the face of significant conflict in certain contexts as pointed out in [34]. Fortunately, in the context of the aggregated assessment based on a hierarchical evaluation model, by discounting all BPAs m_i $(i = 1, \ldots, L)$ at corresponding rates $(1 - w_i)$ $(i = 1, \ldots, L)$, we actually reduce conflict between the various basic assessments before combining them.

Note further that, by definition, focal elements of each $m_i^{w_i}$ are either singleton sets or the whole set $\mathcal{H}$. It is easy to see that m also verifies this property if applicable. Interestingly, the commutative and associative properties of Dempster's rule of combination with respect to a combinable collection of BPAs $m_i^{w_i}$ $(i = 1, \ldots, L)$ and the mentioned property essentially form the basis for the ER algorithms developed in [28], [33]. In other words, the original ER algorithm summarized in (9)–(10) has been implemented for calculation of the BPA m. More particularly, with the same notations as in preceding section, and denoting further

$$m_{I(i)} = \bigoplus_{j=1}^{i} m_j^{w_j}$$

for $i = 1, \ldots, L$, we have

$$m_{I(i)}(H_n) = m_{n,I(i)}, \text{ for } n = 1, \ldots, N \tag{27}$$

$$m_{I(i)}(\mathcal{H}) = m_{\mathcal{H},I(i)} \tag{28}$$

Further, by a simple induction, we easily see that the following holds

Lemma 1: With the quantity $\overline{m}_{\mathcal{H},I(i)}$ inductively defined by (19), we have

$$\overline{m}_{\mathcal{H},I(i)} = K_{I(i)} \prod_{j=1}^{i} (1 - w_j) \tag{29}$$

where $K_{I(i)}$ is inductively defined by (11).

By now, it is obviously clear that, except the weight normalization, the key difference between the original ER algorithm and the modified ER algorithm is nothing but the way of assignment of β_n $(n = 1, \ldots, N)$ and $\beta_{\mathcal{H}}$ after obtained m. That is, in the original ER algorithm, the BPA m is directly used to define the assessment for y by assigning

$$\beta_n = m(H_n) = m_{n,I(L)}, \text{ for } n = 1, \ldots, N \tag{30}$$

$$\beta_{\mathcal{H}} = m(\mathcal{H}) = m_{\mathcal{H},I(L)} \tag{31}$$

While in the modified ER algorithm, after obtained the BPA m, instead of using m to define the assessment for y as in the original ER algorithm, it defines a BPA m' derived from m as follows

$$m'(H_n) = \frac{m(H_n)}{1 - \overline{m}_{\mathcal{H},I(L)}}, \text{ for } n = 1, \ldots, N \tag{32}$$

$$m'(\mathcal{H}) = \frac{(m(\mathcal{H}) - \overline{m}_{\mathcal{H},I(L)})}{1 - \overline{m}_{\mathcal{H},I(L)}} = \frac{\tilde{m}_{\mathcal{H},I(L)}}{1 - \overline{m}_{\mathcal{H},I(L)}} \tag{33}$$

Note that in this case we must have $w_i < 1$ for all $i = 1, \ldots, L$.

Then the assessment for y is defined by assigning

$$\beta_n = m'(H_n), \text{ for } n = 1, \ldots, N \tag{34}$$

$$\beta_{\mathcal{H}} = m'(\mathcal{H}) \tag{35}$$

By (32)–(33), Theorem 2 straightforwardly follows as m is a BPA. Further, the following lemma holds.

Lemma 2: If all assessments $S(e_i)$ $(i = 1, \dots, L)$ are complete, we have

$$m(\mathcal{H}) = \overline{m}_{\mathcal{H},I(L)} = K_{I(L)} \prod_{i=1}^{L} (1 - w_i) \tag{36}$$

i.e., $\tilde{m}_{\mathcal{H},I(L)} = 0$; and, consequently, $S(y)$ defined by (34) is also complete.

As if $w_i = 0$ then the BPA $m_i^{w_i}$ immediately becomes the *vacuous* BPA, and, consequently, plays no role in the aggregation. Thus, without any loss of generality, we assume that $0 < w_i < 1$ for all $i = 1, \dots, L$. Under this assumption, we are easily to see that if the assumption of Theorem 4 holds, then

$$\mathcal{F}_{m_i^{w_i}} = \{\{H_n\} | n \in I^+\} \cup \{\mathcal{H}\}, \text{ for } i = 1, \dots, L \tag{37}$$

where $\mathcal{F}_{m_i^{w_i}}$ denotes the family of focal elements of $m_i^{w_i}$. Hence, by a simple induction, we also have

$$\mathcal{F}_m = \{\{H_n\} | n \in I^+\} \cup \{\mathcal{H}\} \tag{38}$$

Note that the assumption of Theorem 3 is the same as that given in Theorem 4 with $|I^+| = 1$.

Therefore, Theorems 3 and 4 immediately follows from Lemma 2 along with (32)–(35) and (38).

It is also easily seen that

$$m(\mathcal{H}) = K_{I(L)} \prod_{i=1}^{L} m_i^{w_i}(\mathcal{H}) = K_{I(L)} \prod_{i=1}^{L} [w_i m_i(\mathcal{H}) + (1 - w_i)] \tag{39}$$

and in addition, if there is an incomplete assessment, say $S(e_j)$, then $w_j m_j(\mathcal{H}) > 0$, resulting in

$$w_j m_j(\mathcal{H}) \prod_{\substack{i=1 \\ i \neq j}}^{L} (1 - w_i) > 0$$

This directly implies $m'(\mathcal{H}) > 0$. Consequently, Theorem 4 follows as (34)–(35).

B. *The Discounting-and-Yager's Combination Scheme*

To address the issue of conflict as mentioned above, Yager proposed in [24] a modification of Dempster's rule of combination by adding the total probability mass associated with conflict to the frame of discerment, instead of using it for normalization. That is, given two BPAs m_1 and m_2 over a frame Θ, Yager's rule of combination yields a BPA denoted by m^Y as shown in (40) below.

As such, in Yager's rule of combination, the total probability mass associated with conflict between the two BPAs to be combinated is attributed to the frame Θ and, consequently, enlarges the degree of ignorance.

In the context of multi-attribute assessment framework, after discounted the BPA m_i $(i = 1, \dots, L)$ obtained from a basic assessment for e_i at a discount rate of $(1 - w_i)$, we would now like to apply Yager's rule of combination for obtaining an aggregated BPA for the assessment of the general attribute y. As Yager's rule of combination is not associative, we can not combine $m_i^{w_i}$ $(i = 1, \dots, L)$ in a recursive manner as the case of Dempster's rule, but apply a multiple-piece of evidence version defined in [24]. This rule is suitable for an aggregation process (but not a updating process) as in the multiple attribute aggregation.

Particularly, we define m^Y as a combination of BPAs $m_i^{w_i}$ $(i = 1, \dots, L)$ as shown in (41) at the bottom of the page.

Recall, by definition, that focal elements of each $m_i^{w_i}$ are either singleton sets or the whole set $\mathcal{H}$. For $i = 1, \dots, L$, let us denote

$$\mathcal{F}_i = \{\{H_n\} | H_n \in \mathcal{H} \wedge w_i \beta_{n,i} > 0\} \cup \{\mathcal{H}\} \tag{42}$$

With this notation, we have the family of focal elements of $m_i^{w_i}$ $(i = 1, \dots, L)$ is $\mathcal{F}_i^+ = \mathcal{F}_i \setminus \{\mathcal{H}\}$ if $w_i = 1$ and $S(e_i)$ is complete, otherwise $\mathcal{F}_i$ is. For simplicity, we use H_n instead of $\{H_n\}$ without danger of confusion.

Then, we get

$$m^Y(H_n) = \sum_{\substack{H^i \in \mathcal{F}_i \\ \bigcap_{i=1}^{L} H^i = H_n}} \prod_{i=1}^{L} m_i^{w_i}(H^i) \tag{43}$$

$$m_1 \oplus m_2(A) \triangleq m^Y(A) = \begin{cases} 0, & \text{if } A = \emptyset \\ m_1(\Theta)m_2(\Theta) + \sum_{B \cap C = \emptyset} m_1(B)m_2(C), & \text{if } A = \Theta \\ \sum_{B \cap C = A} m_1(B)m_2(C), & \text{otherwise} \end{cases} \tag{40}$$

$$m^Y(H) = \begin{cases} 0, & \text{if } H = \emptyset \\ \prod_{i=1}^{L} m_i^{w_i}(\mathcal{H}) + \sum_{\substack{H^i \subseteq \mathcal{H} \\ \bigcap_{i=1}^{L} H^i = \emptyset}} \prod_{i=1}^{L} m_i^{w_i}(H^i), & \text{if } H = \mathcal{H} \\ \sum_{\substack{H^i \subseteq \mathcal{H} \\ \bigcap_{i=1}^{L} H^i = H}} \prod_{i=1}^{L} m_i^{w_i}(H^i), & \text{otherwise} \end{cases} \tag{41}$$

$$m^Y(\mathcal{H}) = \prod_{i=1}^{L} m_i^{w_i}(\mathcal{H}) + K \quad (44)$$

where K is a constant defined as

$$K = \sum_{\substack{H^i \in \mathcal{F}_i^+ \\ \bigcap_{i=1}^{L} H^i = \emptyset}} \prod_{i=1}^{L} m_i^{w_i}(H^i) \quad (45)$$

representing the total degree of conflict.

Now, by this aggregation scheme we can define

$$\beta_n = m^Y(H_n), \text{ for } n = 1, \ldots, N \quad (46)$$

$$\beta_{\mathcal{H}} = m^Y(\mathcal{H}) = 1 - \sum_{n=1}^{N} m^Y(H_n) \quad (47)$$

for generating the aggregated assessment for y.

Let us denote

$$I = \{1, 2, \ldots, L\} \quad (48)$$

$$I_n^+ = \{i \in I | w_i \beta_{n,i} > 0\}, \text{ for } n = 1, \ldots, N \quad (49)$$

and $\mathcal{P}(I_n^+)$ the power set of $\mathcal{I}_n^+$. Then β_n $(n = 1, \ldots, N)$ is calculated algorithmically as shown in (50) at the bottom of the page.

We are now ready to concern with the synthesis axioms. Obviously, the first independency axiom is followed as there is at least one H^i in (43) being H_n, thus, we have $\beta_n = 0$ if $\beta_{n,i} = 0$ for all i. Similar to the original ER algorithm, we have the following

Theorem 6: If β_n and $\beta_{\mathcal{H}}$ are calculated using (46) and (47), then the concensus axiom holds if and only if

$$\prod_{i=1}^{L}(1 - w_i) = 0$$

Proof: Assume that $\beta_{k,i} = 1$ for all $i = 1, \ldots, L$, and $\beta_{n,i} = 0$ for $n = 1, \ldots, N$, $n \neq k$, and $i = 1, \ldots, L$.

By definition, we have

$$m_i^{w_i}(H_n) = \begin{cases} w_i, & \text{if } n = k \\ 0, & \text{if } n \neq k \end{cases}$$

and $m_i^{w_i}(\mathcal{H}) = (1 - w_i)$, for $i = 1, \ldots, L$. Consequently, $\mathcal{F}_i^+ = \{H_k\}$. This directly implies from (45) that $K = 0$. So, we obtain

$$\beta_{\mathcal{H}} = \prod_{i=1}^{L}(1 - w_i)$$

Then, the consensus axiom and that $\mathcal{F}_i^+ = \{H_k\}$ immediately imply

$$\beta_{\mathcal{H}} = \prod_{i=1}^{L}(1 - w_i) = 0$$

Conversely, if $\prod_{i=1}^{L}(1 - w_i) = 0$ then the consensus axiom is trivially satisfied. This concludes the proof. ■

Unsurprisingly, as in case of the discounting-and-orthogonal sum scheme above, the discounting-and-Yager's combination scheme does not directly yield a generated assessment for y exactly satisfying the completeness axiom. This can be overcome by modifying the assignment of β_n's and $\beta_{\mathcal{H}}$ from m^Y as shown in the following.

C. The Modified Discounting-and-Yager's Combination Scheme

As we have seen, the direct use of the discounting-and-Yager's combination scheme for defining the assessment for y makes it fail to desiredly satisfy the synthesis axioms. This is caused mainly by the fact that an *aggregated rate of discount* is attributed to $\beta_{\mathcal{H}}$ as a part of the unassigned probability mass. Yet a so-called *degree of disagreement* as a part of the conflict factor K also plays a role. In this part, instead of commiting these factors to the unassigned probability mass, they are used for the normalization before assigning for β_n's and $\beta_{\mathcal{H}}$. However, before doing so, we must first be clear on what these factors are.

Denote

$$\mathcal{F}^+ = \bigcap_{i=1}^{L} \mathcal{F}_i^+ \quad (51)$$

where $\mathcal{F}_i^+ = \mathcal{F}_i \setminus \{\mathcal{H}\}$ $(i = 1, \ldots, L)$. That is, $\mathcal{F}^+$ consists of common singleton focal elements of all BPAs $m_i^{w_i}$'s. In other words, all basic attributes e_i $(i = 1, \ldots, L)$ are assessed to all evaluation grades in $\mathcal{F}^+$ to variously positive degrees of belief. As a part of conflict arose during the aggregation process of basic assessments, we define the degree of disagreement, denoted by κ_1, among the various basic assessments at evaluation grades in $\mathcal{F}^+$ as follows

$$\kappa_1 = \sum_{\substack{H^i \in \mathcal{F}^+ \\ \bigcap_{i=1}^{L} H^i = \emptyset}} \prod_{i=1}^{L} m_i^{w_i}(H^i) \quad (52)$$

Note that κ_1 is also a constant and is a part of the degree of conflict K defined by (45) as shown in Appendix.

Due to the multiplicative nature of the combination rule, we define the aggregated rate of discount, denoted by κ_2, as

$$\kappa_2 = \prod_{i=1}^{L}(1 - w_i) \quad (53)$$

Also, κ_2 is a constant and is a part of $\prod_{i=1}^{L} m_i^{w_i}(\mathcal{H})$.

The assignment of a constant amount of $(\kappa_1 + \kappa_2)$ to $\beta_{\mathcal{H}}$ as part of unassigned probability mass may cause the aggregated

$$m^Y(H_n) = \sum_{\emptyset \neq \tau \in \mathcal{P}(I_n^+)} \prod_{i \in \tau} w_i \beta_{n,i} \prod_{j \in I \setminus \tau} \left[w_j \left(1 - \sum_{n=1}^{N} \beta_{n,j}\right) + (1 - w_j) \right] \quad (50)$$

assessment to be incomplete even when all basic assessments are not. Therefore, in the modified discounting-and-Yager's combination scheme, this amount of $(\kappa_1 + \kappa_2)$ is assigned proportionally back to all individual grades and $\mathcal{H}$ using the following normalization process

$$\beta_n = \frac{m^Y(H_n)}{1-(\kappa_1+\kappa_2)}, \text{ for } n = 1, \dots, N \tag{54}$$

$$\beta_{\mathcal{H}} = \frac{m^Y(\mathcal{H}) - (\kappa_1+\kappa_2)}{1-(\kappa_1+\kappa_2)} \tag{55}$$

As remarked above on κ_1 and κ_2, it is easily seen, by definition, that the following holds.

Proposition 1: The degrees of belief generated using (54)–(55) satisfy the following

$$0 \le \beta_n, \beta_{\mathcal{H}} \le 1, \quad \text{for } n = 1, \dots, N$$
$$\sum_{n=1}^{N} \beta_n + \beta_{\mathcal{H}} = 1$$

Regarding the synthesis axioms, we have desiredly the following.

Theorem 7: The aggregated assessment for y defined as in (54)–(55) exactly satisfies all four synthesis axioms.

Proof: See Appendix. ■

D. The Discounting-and-Averaging Scheme

In the aggregation schemes above, we have defined, for each $i = 1, \dots, L$, $(w_i \otimes m_i)$ as the BPA $m_i^{w_i}$ by discounting m_i by a factor $(1 - w_i)$ (refer to (25)). Then Dempster's and Yager's rules of combination are respectively applied for obtaining the BPA m of the assessment for y. Here, we also assume that $0 < w_i \le 1$ for all $i = 1, \dots, N$.

In this subsection, instead of applying these combination operations after discounting m_i's, we apply the averaging operation over L BPAs $m_i^{w_i}$ ($i = 1, \dots, L$) to obtain a BPA $\overline{m}$ defined by

$$\overline{m}(H) = \frac{1}{L}\sum_{i=1}^{L} m_i^{w_i}(H) \tag{56}$$

for any $H \subseteq \mathcal{H}$.

Due to (25), we have

$$\overline{m}(H) = \begin{cases} \frac{1}{L}\sum\limits_{i=1}^{L} w_i \beta_{n,i}, & \text{if } H = \{H_n\} \\ \frac{1}{L}\sum\limits_{i=1}^{L}\left(1 - w_i \sum\limits_{n=1}^{N}\beta_{n,i}\right), & \text{if } H = \mathcal{H} \\ 0, & \text{otherwise} \end{cases} \tag{57}$$

After obtaining the aggregated BPA $\overline{m}$, the problem now is to use $\overline{m}$ for generating the aggregated assessment for the general attribute y. Naturally, we can assign

$$\beta_n = \overline{m}(H_n) = \frac{1}{L}\sum_{i=1}^{L} w_i \beta_{n,i}, \text{ for } n = 1, \dots, N \tag{58}$$

$$\beta_{\mathcal{H}} = \overline{m}(\mathcal{H}) = \frac{1}{L}\sum_{i=1}^{L}\left(1 - w_i \sum_{n=1}^{N}\beta_{n,i}\right) \tag{59}$$

Then the assessment for y is defined by

$$S(y) = \{(H_n, \beta_n) | n = 1, \dots, N\} \tag{60}$$

Regarding the synthesis axioms, we easily see that the first axiom holds for the assessment (60). For the next two axioms, we have the following

Theorem 8: The assessment (60) defined via (58)–(59) satisfies the consensus axiom and/or the completeness axiom if and only if $w_i = 1$ for all $i = 1, \dots, L$.

Proof: For the consensus axiom, the proof is straightforward. Now we rewrite $\beta_{\mathcal{H}}$ defined by (59) as follows

$$\begin{aligned} \beta_{\mathcal{H}} &= \frac{1}{L}\sum_{i=1}^{L}\left(1 - w_i \sum_{n=1}^{N}\beta_{n,i}\right) \\ &= \frac{1}{L}\sum_{i=1}^{L} w_i \left(1 - \sum_{n=1}^{N}\beta_{n,i}\right) + \left(1 - \frac{\sum_{i=1}^{L} w_i}{L}\right) \end{aligned} \tag{61}$$

Thus, if all $w_i = 1$ and the assumption of the completeness axiom holds, we have $\beta_{\mathcal{H}} = 0$ and the conclusion of the axiom follows easily. Inversely, if the completeness axiom is satisfied, we must have

$$1 - \frac{\sum\limits_{i=1}^{L} w_i}{L} = 0$$

which directly implies $w_i = 1$ for all i. ■

The assessment for y according to this aggregation scheme also satisfies the incompleteness axiom trivially due to the nature of discounting-and-averaging.

Unfortunately, the requirement of $w_i = 1$ for all i to satisfy the consensus axiom and the completeness axiom would not be appropriate in general. This is due to the allocation of the average of discount rates

$$\overline{\alpha} \triangleq \left(1 - \frac{\sum\limits_{i=1}^{L} w_i}{L}\right)$$

to $\mathcal{H}$ as a part of unassigned probability mass. This dilemma can be resolved in a similar way as in the modified algorithms above. Interestingly, this modification leads to the weighted sum scheme as shown in the following.

E. Weighted Sum as the Modified Discounting-and-Averaging Scheme

By applying the discounting-and-averaging scheme, we obtain the BPA $\overline{m}$ as defined by (57). Now, guided by the synthesis axioms, instead of making direct use of $\overline{m}$ in defining the generated assessment $S(y)$ (i.e., allocating the average discount rate $\overline{\alpha}$ to $\beta_{\mathcal{H}}$ as a part of unassigned probability mass) as above, we define a new BPA denoted by $\overline{m}'$ derived from $\overline{m}$ by making use of $(1 - \overline{\alpha})$ as a normalization factor. More particularly, we define

$$\overline{m}'(H_n) = \frac{\overline{m}(H_n)}{1-\overline{\alpha}}, \text{ for } n = 1, \dots, N \tag{62}$$

$$\overline{m}'(\mathcal{H}) = \frac{\overline{m}(\mathcal{H}) - \overline{\alpha}}{1-\overline{\alpha}} \tag{63}$$

Combining (57) and (61) we interestingly obtain

$$\overline{m}'(H_n) = \sum_{i=1}^{L} \overline{w}_i \beta_{n,i}, \text{ for } n = 1, \ldots, N \tag{64}$$

$$\overline{m}'(\mathcal{H}) = \sum_{i=1}^{L} \overline{w}_i \left(1 - \sum_{n=1}^{N} \beta_{n,i}\right) \tag{65}$$

where

$$\overline{w}_i = \frac{w_i}{\sum_{i=1}^{L} w_i}, \text{ for } i = 1, \ldots, L$$

Let us turn back to the general scheme of combination given in (24). Under the view of this general scheme, the above BPA $\overline{m}'$ is nothing but an instance of it by simply considering $\otimes$ as the multiplication and $\oplus$ as the weighted sum. Namely, we have

$$\overline{m}'(H_n) = \sum_{i=1}^{L} \overline{w}_i m_i(H_n), \text{ for } n = 1, \ldots, N \tag{66}$$

$$\overline{m}'(\mathcal{H}) = \sum_{i=1}^{L} \overline{w}_i m_i(\mathcal{H}) \tag{67}$$

where relative weights $\overline{w}_i$ are normalized as above so that $\sum_i \overline{w}_i = 1$. It is of interest to note that the possibility of using such an operation has previously been mentioned in, for example, [25]. Especially, the weighted sum operation of two BPAs has been used for the integration of distributed databases for purposes of data mining [12].

Now we quite naturally define the assessment for y by assigning

$$\beta_n = \overline{m}'(H_n) = \sum_{i=1}^{L} \overline{w}_i m_i(H_n), \text{ for } n = 1, \ldots, N \tag{68}$$

$$\beta_{\mathcal{H}} = \overline{m}'(\mathcal{H}) = \sum_{i=1}^{L} \overline{w}_i m_i(\mathcal{H}) \tag{69}$$

Appealingly simple as it is, we can see quite straightforwardly that the following theorem holds.

Proposition 2: The degrees of belief generated using (68)–(69) satisfy the following

$$0 \le \beta_n, \beta_{\mathcal{H}} \le 1, \quad \text{for } n = 1, \ldots, N$$

$$\sum_{n=1}^{N} \beta_n + \beta_{\mathcal{H}} = 1$$

Furthermore, concerning the synthesis axioms, we have the following theorem.

Theorem 9: The aggregated assessment for y defined as in (68)–(69) exactly satisfies all four synthesis axioms.

Proof: Trivially. ■

F. Expected Utility in the ER Approaches

In the tradition of decision making under uncertainty [17], the notion of expected utility has been mainly used to rank alternatives in a particular problem. That is one can represent the preference relation $\succeq$ on a set of alternatives X with a single-valued function $u(x)$ on X, called *expected utility*, such that for any $x, y \in X$, $x \succeq y$ if and only if $u(x) \ge u(y)$. Maximization of $u(x)$ over X provides the solution to the problem of selecting x.

In the ER approach, we assume a utility function

$$u' : \mathcal{H} \to [0, 1]$$

satisfying

$$u'(H_{n+1}) > u'(H_n) \text{ if } H_{n+1} \text{ is preferred to } H_n.$$

This utility function u' may be determined using the probability assignment method [10] or using other methods as in [28], [33].

If all assessments for basic attributes are complete, Lemma 2 shows that the assessment for y is also complete, i.e. $\beta_{\mathcal{H}} = 0$. Then the expected utility of an alternative on the attribute y is defined by

$$u(y) = \sum_{n=1}^{N} \beta_n u'(H_n) \tag{70}$$

An alternative a is strictly preferred to another alternative b if and only if $u(y(a)) > u(y(b))$.

Due to incompleteness, in general, in basic assessments, the assessment for y may result in incomplete. In such a case, in [33] the authors define three measures, called the minimum, maximum and average expected utilities, as follows

$$u_{\max}(y) = \sum_{n=1}^{N-1} \beta_n u'(H_n) + (\beta_N + \beta_{\mathcal{H}}) u'(H_n) \tag{71}$$

$$u_{\min}(y) = (\beta_1 + \beta_{\mathcal{H}}) u'(H_1) + \sum_{n=2}^{N} \beta_n u'(H_n) \tag{72}$$

$$u_{\text{avg}}(y) = \frac{u_{\max}(y) + u_{\min}(y)}{2} \tag{73}$$

where, without loss of generality, suppose H_1 is the least preferred grade having the lowest utility and H_N the most preferred grade having the highest utility.

The ranking of two alternatives a and b on y is carried out by:

- $a \succ_y b$ if and only if $u_{\min}(y(a)) > u_{\max}(y(b))$
- $a \sim_y b$ if and only if $u_{\min}(y(a)) = u_{\min}(y(b))$ and $u_{\max}(y(a)) = u_{\max}(y(b))$

If these are not the case, the average expected utility can be used to generate a ranking (see, e.g., [33] for more details).

In this paper, based on the *Generalized Insufficient Reason Principle*, we define a probability function P_m on $\mathcal{H}$ derived from m for the purpose of making decisions via the *pignistic transformation* [21]. Namely,

$$P_m(H_n) = m(H_n) + \frac{1}{N} m(\mathcal{H}) \text{ for } n = 1, \ldots, N \tag{74}$$

That is, as in the two-level language of the so-called *transferable belief model* [21], the aggregated BPA m itself represented the belief is entertained based on the available evidence at the *credal level*, and when a decision must be made, the belief at the credal level induces the probability function P_m defined by (74) for decision making. Particularly, the approximately assessment for y for the purpose of decision making is then defined as

$$\beta'_n \quad = \quad P_m(H_n) = \beta_n + \frac{1}{N}\beta_{\mathcal{H}}, \text{ for } n = 1, \ldots, N \quad (75)$$

Therefore, the expected utility of an alternative on the attribute y is straightforwardly defined by

$$u(y) = \sum_{n=1}^{N} \beta'_n u'(H_n) = \sum_{n=1}^{N} (\beta_n + \frac{1}{N}\beta_{\mathcal{H}}) u'(H_n) \qquad (76)$$

As such, while the amount of belief $\beta_{\mathcal{H}}$ (due to ignorance) is allocated either to the least preferred grade H_1 or to the most preferred grade H_N to define the expected utility interval in Yang's approach [33], in our approach it is uniformly allocated to every evaluation grade H_n, guided by the Generalized Insufficient Reason Principle [21], to define an approximately assessment for y and, hence, a single-valued expected utility function.

In the following section, we examine a turorial example taken from [33] to figure out how difference between the various aggregation schemes as well as the respective results yielded.

V. An Example: Motorcycle Assessment Problem

The problem is to evaluate the performance of four types of motorcycles, namely *Kawasaki*, *Yamaha*, *Honda*, and *BMW*.

The overall performance of each motorcycle is evaluated based on three major attributes which are *quality of engine*, *operation*, *general finish*. These attributes all are general and difficult to assess directly. So these attributes are correspondingly decomposed into more detailed subattributes to facilitate the assessment. The process of attribute decomposition for the evaluation problem of motorcycles results in an attribute hierarchy graphically depicted in Fig. 3, where the relative weights of attributes at a single level associated with the same upper level attribute are defined by w_i, w_{ij}, and w_{ijk} for the attributes at level 1, 2, and 3 respectively.

Using the five-grade evaluation scale as given in (1), the assessment problem of motorcycles is given in Table I, where P, I, A, G, and E are the abbreviations of *poor*, *indifferent*, *average*, *good*, and *excellent*, respectively, and a number in bracket denoted the degree of belief to which an attribute is assessed to a grade. For example, $E(0.8)$ means "*excellent* to a degree of 0.8".

Further, all relevant attributes are assumed to be of equal relative important [33]. That is

$$\begin{aligned}
w_1 = w_2 = w_3 &= 0.3333 \\
w_{11} = w_{12} = w_{13} = w_{14} = w_{15} &= 0.2 \\
w_{21} = w_{22} = w_{23} &= 0.3333 \\
w_{211} = w_{212} = w_{213} = w_{214} &= 0.25 \\
w_{221} = w_{222} &= 0.5 \\
w_{231} = w_{232} = w_{233} &= 0.3333 \\
w_{31} = w_{32} = w_{33} = w_{34} = w_{35} &= 0.2
\end{aligned}$$

In the sequent, for the purpose of comparison, we generate different three results of aggregation corresponding to Yang and Xu's modified ER method and the other two developed in this paper.

By applying the modified ER method, the distributed assessments for overall performance of four types of motorcycles are given in Table II. These four distrubutions and their approximations via the pignistic transformation (Table III) are graphically shown as in Fig. 4.

At the same time, by applying the weighted sum aggregation scheme (shortly, SW method), we easily obtain the distributed assessments for overall performance of four types of motorcycles as shown in Table IV (graphically depicted in Fig. 5 (a)). The pignistic transformation applied to these aggregated assessments yields the approximately assessments for overall performance of motorcycles as given in Table V (graphically, Fig. 5 (b)).

As we can easily see, there is not so much difference between the result obtained by the modified ER algorithm and that obtained by the weighted sum method, especially the behavior of correspondingly assessment distributions is almost the same as Fig. 4 (a) and Fig. 5 (a) have shown.

However, as we see in the following, the result yielded by the modified Yager's combination method (shortly, MY method) is relatively different from those obtained by the above methods. This is unsurprising as we were attributing a factor of conflict to $\mathcal{H}$ as "unknown" in the aggregated assessment.

For generating the assessment for an attribute y at a higher level in the hierarchy of attributes shown in Fig. 3, all the BPAs of its direct subattributes are firstly aggregated via (50), and the generated assessment for y is then obtained using the normalization process represented in (54) and (55). This process is carried out upward from the bottom level to the top of the hierarchy in order to obtain the overall assessment. With this method of aggregation, we obtain the distributed assessments for overall performance of four types of motorcycles as shown in Table VII, which are graphically depicted in Fig. 6 (a).

From the obtained result, it is interesting to observe that, although a total degree of incompleteness in basic assessments of *Honda* is 1.25 in compare to those of the other three, which in turn are 0.5 for both *Kawasaki* and *Yamaha*, and 0.4 for *BMW*, the unassigned probability mass of the generated assessment for *Honda* is smaller than those of the remainders. This is due to a lower conflict between basic assessments of *Honda* in compare to those of the others.

For the purpose of decision making, we apply the pignistic transformation to the aggregated assessments in order to obtain the approximately assessments for overall performance of motorcycles as shown in Table VII and depicted graphically in Fig. 5 (b).

We are now ready to assume a utility function $u' : \mathcal{H} \rightarrow [0, 1]$ defined in [33] as follows

$$u'(P) = 0, \quad u'(I) = 0.35,$$
$$u'(A) = 0.55, u'(G) = 0.85, u'(E) = 1$$

Using (76), we easily obtain the expected utility of four types of motorcycles according to the various methods as given in Table VIII.

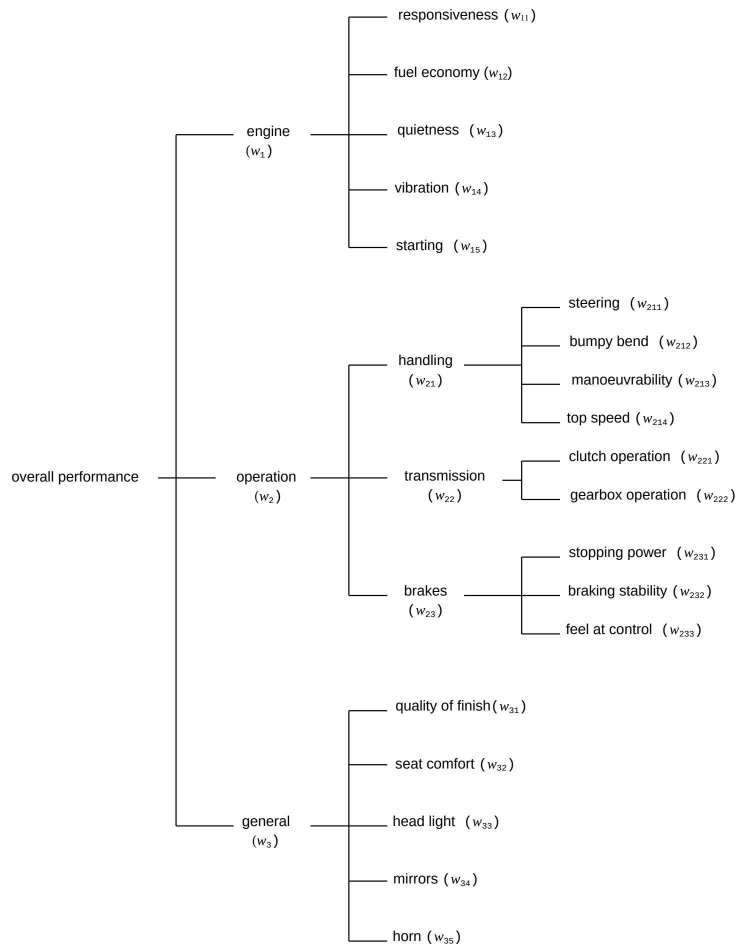


Fig. 3. Evaluation hierarchy for motorcycle performance assessment [33]

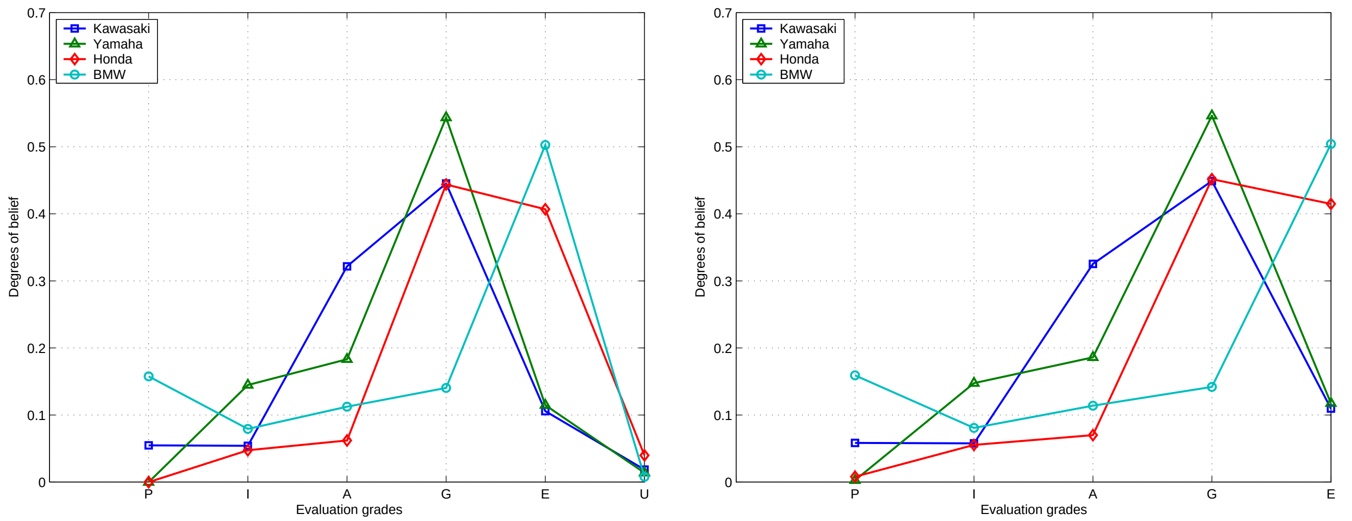
TABLE I
GENERALIZED DECISION MATRIX FOR MOTORCYCLE ASSESSMENT [33]

General attributes			**Basic attributes**	**types of motor cycle (alternatives)**			
				Kawasaki (a_1)	*Yamaha* (a_2)	*Honda* (a_3)	*BMW* (a_4)
Overall performance	engine		responsiveness	E (0.8)	G (0.3) E (0.6)	G (1.0)	I(1.0)
			fuel economy	A(1.0)	I(1.0)	I(0.5) A(0.5)	E(1.0)
			quietness	I(0.5) A(0.5)	A(1.0)	G(0.5) E(0.3)	E(1.0)
			vibration	G (1.0)	I(1.0)	G(0.5) E(0.5)	P(1.0)
			starting	G (1.0)	A(0.6) G(0.3)	G (1.0)	A(1.0)
	operation	handling	steering	E(0.9)	G(1.0)	A(1.0)	A(0.6)
			bumpy bends	A(0.5) G(0.5)	G (1.0)	G(0.8) E(0.1)	P(0.5) I(0.5)
			maneuverability	A(1.0)	E(0.9)	I(1.0)	P(1.0)
			top speed stability	E(1.0)	G(1.0)	G(1.0)	G(0.6) E(0.4)
		transmission	clutch operation	A(0.8)	G(1.0)	E(0.85)	I(0.2) A(0.8)
			gearbox operation	A(0.5) G(0.5)	I(0.5) A(0.5)	E(1.0)	P(1.0)
		brakes	stopping power	G(1.0)	A(0.3) G(0.6)	G(0.6)	E(1.0)
			braking stability	G(0.5) E(0.5)	G(1.0)	A(0.5) G(0.5)	E(1.0)
			feel at control	P(1.0)	G(0.5) E(0.5)	G(1.0)	G(0.5) E(0.5)
	general		quality of finish	P(0.5) I(0.5)	G(1.0)	E(1.0)	G(0.5) E(0.5)
			seat comfort	G(1.0)	G(0.5) E(0.5)	G(0.6)	E(1.0)
			headlight	G(1.0)	A(1.0)	E(1.0)	G(0.5) E(0.5)
			mirrors	A(0.5) G(0.5)	G(0.5) E(0.5)	E(1.0)	G(1.0)
			horn	A(1.0)	G(1.0)	G(0.5) E(0.5)	E(1.0)

TABLE II

AGGREGATED ASSESSMENTS FOR FOUR TYPES OF MOTOCYCLES USING THE MODIFIED ER METHOD [33]

	Poor(P)	*Indifference*(I)	*Average*(A)	*Good*(G)	*Excellent*(E)	*Unknown*(U)
Kawasaki	0.0547	0.0541	0.3216	0.4452	0.1058	0.0186
Yamaha	0.0	0.1447	0.1832	0.5435	0.1148	0.0138
Honda	0.0	0.0474	0.0621	0.4437	0.4068	0.0399
BMW	0.1576	0.0792	0.1124	0.1404	0.5026	0.0078



(a) The Aggregated Assessment

(b) Approximation Assessment via Pignistic Transformation

Fig. 4. Overall Evaluation of Motorcycles via the Modified ER Method

TABLE III

APPROXIMATELY ASSESSMENTS FOR FOUR TYPES OF MOTOCYCLES USING THE MODIFIED ER METHOD

	Poor(P)	*Indifference*(I)	*Average*(A)	*Good*(G)	*Excellent*(E)
Kawasaki	0.05842	0.05782	0.32532	0.44892	0.10952
Yamaha	0.00276	0.14746	0.18596	0.54626	0.11756
Honda	0.00798	0.05538	0.07008	0.45168	0.41478
BMW	0.15916	0.08076	0.11396	0.14196	0.50416

TABLE IV

AGGREGATED ASSESSMENTS FOR FOUR TYPES OF MOTOCYCLES BY USING THE WEIGHTED SUM AGGREGATION SCHEME

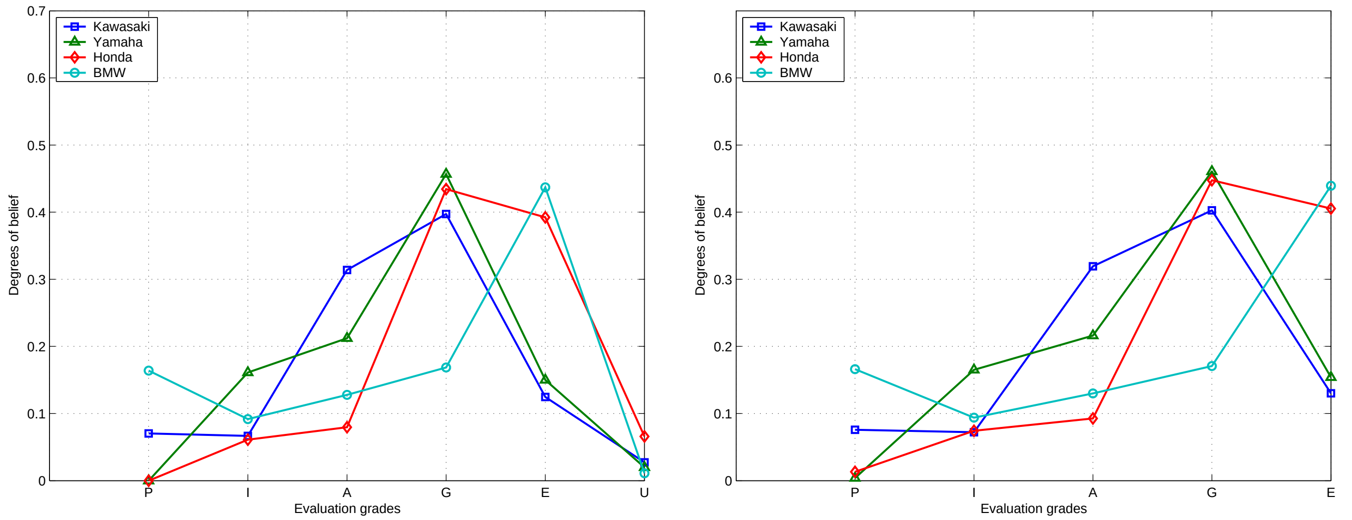
	Poor(P)	*Indifference*(I)	*Average*(A)	*Good*(G)	*Excellent*(E)	*Unknown*(U)
Kawasaki	0.0703	0.0667	0.3139	0.3972	0.1247	0.0272
Yamaha	0.0	0.1611	0.2122	0.4567	0.1501	0.0198
Honda	0.0	0.0611	0.0796	0.4344	0.3922	0.0659
BMW	0.1639	0.0917	0.1278	0.1685	0.437	0.0111

Consequently, the ranking of the four types of motorcycles is given in the Table IX.

Note that the same ranking result for all methods could also be obtained by the expected utility interval and the ranking scheme by Yang and Xu [33] as mentioned above. As we have seen, although the solution to the problem of selecting the best alternative is the same for all the three methods of aggregation, the ranking order between the alternatives is different. More particularly, while *Yamaha* is preferred to *BMW* according to the results of the first two methods, *BMW* is preferred to *Yamaha* according to the third method. This is because, by the third method of aggregation, the former is assessed to *good* and *excellent* to a total degee of 0.5797 while the latter 0.54872.

VI. CONCLUDING REMARKS

In this paper, we have reanalysed the ER approach to MADM under uncertainty. Interestingly, the analysis provides



(a) The Aggregated Assessment (b) Approximation Assessment via Pignistic Transformation

Fig. 5. Overall Evaluation of Motorcycles via Weighted Sum Method

TABLE V

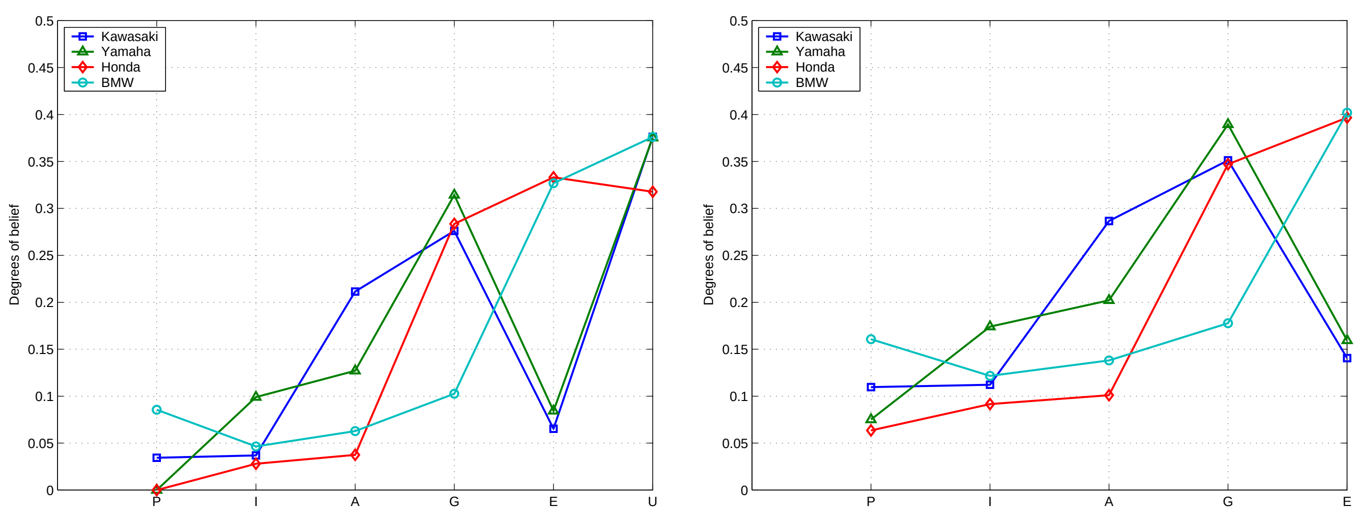
APPROXIMATELY ASSESSMENTS FOR FOUR TYPES OF MOTOCYCLES USING THE WEIGHTED SUM METHOD

	Poor(P)	*Indifference*(I)	*Average*(A)	*Good*(G)	*Excellent*(E)
Kawasaki	0.07574	0.07214	0.31934	0.40264	0.13014
Yamaha	0.00396	0.16506	0.21616	0.46066	0.15406
Honda	0.01318	0.07428	0.09278	0.44758	0.40538
BMW	0.16612	0.09392	0.13	0.17072	0.43922

TABLE VI

AGGREGATED ASSESSMENTS FOR FOUR TYPES OF MOTOCYCLES USING THE MODIFIED YAGER'S COMBINATION METHOD

	Poor(P)	*Indifference*(I)	*Average*(A)	*Good*(G)	*Excellent*(E)	*Unknown*(U)
Kawasaki	0.0344	0.0369	0.2114	0.276	0.0653	0.3761
Yamaha	0	0.099	0.127	0.3143	0.0843	0.3753
Honda	0	0.028	0.0375	0.2835	0.3331	0.3177
BMW	0.0855	0.0464	0.0628	0.1025	0.3268	0.376



(a) The Aggregated Assessment (b) Approximation Assessment via Pignistic Transformation

Fig. 6. Overall Evaluation of Motorcycles via the Modified Yager's Combination Method

TABLE VII

APPROXIMATELY ASSESSMENTS FOR FOUR TYPES OF MOTOCYCLES USING THE MODIFIED YAGER'S COMBINATION METHOD

	Poor(P)	*Indifference*(I)	*Average*(A)	*Good*(G)	*Excellent*(E)
Kawasaki	0.10962	0.11212	0.28662	0.35122	0.14052
Yamaha	0.07506	0.17406	0.20206	0.38936	0.15936
Honda	0.06354	0.09154	0.10104	0.34704	0.39664
BMW	0.1607	0.1216	0.138	0.1777	0.402

TABLE VIII

EXPECTED UTILITY OF FOUR TYPES OF MOTORCYCLES

	Kawasaki	*Yamaha*	*Honda*	*BMW*
ER method	0.69026	0.73577	0.85664	0.71577
WS method	0.67327	0.72228	0.86285	0.68871
MY method	0.63594	0.66237	0.77923	0.67151

TABLE IX

RANKING OF FOUR TYPES OF MOTORCYCLES

Method	**Ranking order**
ER method	*Honda* $\succ$ *Yamaha* $\succ$ *BMW* $\succ$ *Kawasaki*
WS method	*Honda* $\succ$ *Yamaha* $\succ$ *BMW* $\succ$ *Kawasaki*
MY method	*Honda* $\succ$ *BMW* $\succ$ *Yamaha* $\succ$ *Kawasaki*

a general formulation for the attribute aggregation problem in MADM under uncertainty. Under such a generalization, several various aggregation schemes have been examined, including the previous one. Theoretical properties of new schemes regarding the synthesis axioms proposed in [33] were also explored. Especially, by this reformulation of the attribute aggregation problem, we have shown that the aggregation scheme based on the weighted sum operation could be also considered for the aggregation process in the context of MADM under uncertainty. This allows us to handle incomplete uncertain information in a simple and proper manner when the assumption regarding the independence of attributes' uncertain evaluations is not appropriate.

For the purpose of decision making, an approximate method of uncertain assessments based on the so-called pignistic transformation [21] has been applied to define the expected utility function, instead of using the expected utility interval proposed previously. A tutorial example has been examined to illustrate the discussed techniques.

In summary, by the results obtained in this paper, we do hope to support further aggregation schemes for the attribute aggregation problem in MADM under uncertainty. This is especially helpful in decision making situations where a single method of aggregation would be inapplicable or not enough.

APPENDIX

In this Appendix, we give the proof of Theorem 7 on the synthesis axioms for the modified discounting-and-Yager's combination scheme. Clearly, the independency axiom is immediately followed from (43) as the case of the discounting-and-Yager's combination scheme. Note that we assume here that weights w_i's are normalized so that $0 < w_i < 1$ for all $i \in I$. First we need some preparations. Recall that

$$\beta_n = \frac{m^Y(H_n)}{1-(\kappa_1+\kappa_2)}, \text{ for } n = 1,\ldots,N$$
$$\beta_{\mathcal{H}} = \frac{m^Y(\mathcal{H}) - (\kappa_1+\kappa_2)}{1-(\kappa_1+\kappa_2)}$$
$$m^Y(\mathcal{H}) = \Delta + K$$

where

$$\Delta = \prod_{i=1}^{L}[w_i m_i(\mathcal{H}) + (1-w_i)]$$
$$K = \sum_{H^i \in \mathcal{F}_i^+ :\ \bigcap_{i=1}^{L} H^i = \emptyset} \prod_{i=1}^{L} w_i m_i(H^i)$$
$$\mathcal{F}_i^+ = \{H_n | H_n \in \mathcal{H} \wedge w_i \beta_{n,i} > 0\}$$

Let us denote

$$\mathbb{F} = \mathcal{F}_1^+ \times \mathcal{F}_2^+ \times \cdots \times \mathcal{F}_L^+$$
$$\mathbb{F}^+ = \mathcal{F}^+ \times \mathcal{F}^+ \times \cdots \times \mathcal{F}^+$$

where $\times$ denotes the Cartesian product. For

$$H = (H^1, \ldots, H^L) \in \mathbb{F} \quad (\text{ or } \mathbb{F}^+)$$

by $\cap H$ we mean $\cap_{i=1}^{L} H^i$. We can now decompose Δ into two parts as $\Delta = \Delta' + \kappa_2$, where

$$\Delta' = \sum_{\emptyset \neq \tau \in \mathcal{P}(I)} \prod_{i \in \tau} w_i m_i(\mathcal{H}) \prod_{i \in I \setminus \tau} (1-w_i) \quad (77)$$

Similarly, K is decomposed as $K = K' + \kappa_1$ with

$$K' = \sum_{\substack{H \in (\mathbb{F} \setminus \mathbb{F}^+) \\ \cap H = \emptyset}} \prod_{i=1}^{L} w_i m_i(H^i) \quad (78)$$

Proof for the Consensus Axiom

Suppose that $\beta_{k,i} = 1$ for all $i \in I$, and $\beta_{n,i} = 0$ for $k \neq n = 1, \ldots, N$, $n \neq k$, and $i \in I$.

Then, we have

$$m_i^{w_i}(H_n) = \begin{cases} w_i, & \text{if } n = k \\ 0, & \text{if } n \neq k \end{cases}$$

and $m_i^{w_i}(\mathcal{H}) = (1 - w_i)$, for $i \in I$. Thus

$$\mathcal{F}_i^+ = \{H_k\} \text{ for all } i = 1, \ldots, L$$

This directly implies that $K = 0$. Further, we have $\Delta' = 0$, since $m_i(\mathcal{H}) = 0$, for all $i \in I$. Hence, we get

$$m^Y(\mathcal{H}) = \kappa_2 = \prod_{i=1}^{L}(1 - w_i)$$

which immediately follows that $\beta_{\mathcal{H}} = 0$.

Inductively, we have

$$\sum_{\tau \in \mathcal{P}(I)} \prod_{i\in\tau} w_i \prod_{j \in I \setminus \tau} (1 - w_j) = 1$$

Therefore

$$\sum_{\emptyset \neq \tau \in \mathcal{P}(I)} \prod_{i\in\tau} w_i \prod_{j \in I \setminus \tau} (1 - w_j) = 1 - \kappa_2$$

From (49) and the assumption we have $I_k^+ = I$. Thus the last equation and (50) imply that $\beta_k = 1$. This completes the proof for the consensus axiom.

Proof for the Completeness Axiom

Assume $\mathcal{H}^+ \subset \mathcal{H}$ and denote $J^+ = \{n | H_n \in \mathcal{H}^+\}$. We now prove the following statement:

If $\beta_{n,i} > 0$ for $n \in J^+$ and $\sum\limits_{n \in J^+} \beta_{n,i} = 1,$ for all $i \in I$, then $\beta_n > 0$ for $n \in J^+$ and $\sum\limits_{n \in J^+} \beta_n = 1$ as well.

Since $0 < w_i < 1$ for all $i \in I$, we have

$$m_i^{w_i}(H_n) = \begin{cases} w_i\beta_{n,i} > 0, & \text{if } n \in J^+ \\ 0, & \text{otherwise} \end{cases}$$

and hence

$$\mathcal{F}_i^+ = \{H_n | n \in J^+\}, \quad \text{for all } i \in I$$

Therefore, $\mathbb{F}^+ = \mathbb{F}$, which directly follows $K' = 0$. Further, from (49) we get $I_n^+ = I$ for all $n \in J^+$. Using (50), we obtain

$$m^Y(H_n) = \sum_{\emptyset \neq \tau \in \mathcal{P}(I)} \prod_{i\in\tau} w_i\beta_{n,i} \prod_{j \in I \setminus \tau} (1 - w_j) \tag{79}$$

for any $H_n \in \mathcal{H}^+$ (i.e., $n \in J^+$).

On the other hand, we have $m_i(\mathcal{H}) = 0$ for all i. So, we have also $\Delta' = 0$. By definition, we get

$$m^Y(\mathcal{H}) = \kappa_1 + \kappa_2$$

Thus

$$\beta_{\mathcal{H}} = \frac{m^Y(\mathcal{H}) - (\kappa_1 + \kappa_2)}{1 - (\kappa_1 + \kappa_2)} = 0$$

From (43) and the assumption $\beta_{n,i} = 0$ if $n \notin J^+$, for all $i \in I,$ we easily deduce that

$$m^Y(H_n) = 0, \quad \text{for any } n \in \{1, \ldots, N\} \setminus J^+$$

Immediately, it follows

$$\beta_n = 0, \quad \text{for any } n \in \{1, \ldots, N\} \setminus J^+$$

Again, since $0 < w_i < 1$ for all i, it follows $\kappa_2 > 0$. This implies from (79) that $m^Y(H_n) > 0$ for all $n \in J^+$. Thus, $\beta_n > 0$ for all $n \in J^+$. Finally, the desired equation

$$\sum_{n \in J^+} \beta_n = 1$$

is followed as $\sum_{n=1}^N \beta_n + \beta_{\mathcal{H}} = 1$. This concludes the proof for the completeness axiom.

Proof for the Incompleteness Axiom

Now we give a proof for the last axiom. Assume there is an index $i_0 \in I$ such that $\sum\limits_{n=1}^{N} \beta_{n,i_0} < 1,$ we must prove that

$$\sum_{n=1}^{N} \beta_n < 1, \text{ or equivalently, } \beta_{\mathcal{H}} > 0$$

By definition, we have

$$\beta_{\mathcal{H}} = \frac{m^Y(\mathcal{H}) - (\kappa_1 + \kappa_2)}{1 - (\kappa_1 + \kappa_2)} = \frac{\Delta' + K'}{1 - (\kappa_1 + \kappa_2)}$$

So it is sufficient to show either $\Delta' > 0$ or $K' > 0$, say $\Delta' > 0$. Indeed, since

$$\sum_{n=1}^{N} \beta_{n,i_0} < 1$$

we have $m_{i_0}(\mathcal{H}) > 0$. This follows

$$w_{i_0} m_{i_0}(\mathcal{H}) \prod_{i \in I \setminus \{i_0\}} (1 - w_i) > 0$$

as $0 < w_i < 1$ for all $i \in I$. Thus from (77) we easily deduce

$$\Delta' > 0$$

which we desired. This completely concludes the theorem.

Acknowledgment

The work on this paper was partly supported by JAIST 21th COE Project.

References

[1] G. Balestra, A. Tsoukias, "Multicriteria analysis represented by artificial intelligence techniques," *European Journal of Operational Research*, vol. 41, pp. 419–430, 1990.

[2] V. Belton, "Comparison of the analytic hierarchy process and a simple multi-attribute value function," *European Journal of Operational Research*, vol. 26, pp. 7–21, 1986.

[3] V. Belton, T. J. Stewart, *Multiple Criteria Decision Analysis: An Integrated Approach*. Norwell, MA: Kluwer, 2002.

[4] B. G. Buchanan, E. H. Shortliffe, *Rule-Based Expert Systems*. Reading, MA: Addison-Wesley, 1984.

[5] S. J. Chen, C. L. Hwang, *Fuzzy Multiple Attribute Decision Making-Methods and Applications*. Berlin: Springer-Verlag, 1992.

[6] D. Dubois, H. Fargier, P. Perny, H. Prade, "A characterization of generalized concordance rules in multicriteria decision making," *International Journal of Intelligent Systems*, vol. 18, pp. 751–774, 2003.

[7] T. Isitt, "The sports tourers," *Motor Cycle International*, vol. 64, pp. 19–27, 1990.

[8] V. N. Huynh, Y. Nakamori, "A satisfactory-oriented approach to multi-expert decision-making under linguistic assessments," submitted to *IEEE Transactions on Systems, Man, and Cybernetics*.

[9] C. L. Hwang and K. Yoon, *Multiple Attribute Decision Making Methods and Applications*. New York: Springer-Verlag, 1981.

[10] R. L. Keeney, H. Raiffa, *Decisions with Multiple Objectives: Preferences and Value Tradeoffs*. New York: John Wiley and Sons, 1976.

[11] L. Levrat, A. Voisin, S. Bombardier, J. Bremont, "Subjective evaluation of car seat comfort with fuzzy set techniques," *International Journal of Intelligent Systems*, vol. 12, pp. 891–913, 1997.

[12] S. McClean, B. Scotney, "Using evidence theory for the integration of distributed databases," *International Journal of Intelligent Systems*, vol. 12, pp. 763–776, 1997.

[13] G. A. Miller, "The magical number seven or minus two: some limits on our capacity of processing information," *The Psychological Review*, vol. 63, pp. 81–97, 1956.

[14] R. Roberts, P. Goodwin, "Weight approximations in multi-attribute decision models," *Journal of Multi-Criteria Decision Analysis*, vol. 11, pp. 291–303, 2002.

[15] M. Roubens, "Fuzzy sets and decision analysis," *Fuzzy Sets and Systems*, vol. 90, pp. 199–206, 1997.

[16] T. L. Saaty, *The Analytic Hierarchy Process*. New York: McGraw-Hill, 1980.

[17] L. J. Savage, *The Foundations of Statistics*. New York: John Wiley and Sons, 1954.

[18] P. Sen, J. B. Yang, "Multiple criteria decision making in design selection and synthesis," *Journal of Engineering Design*, vol. 6, pp. 207–230, 1995.

[19] K. Sentz, S. Ferson, "Combination of evidence in Dempster-Shafer theory," *SAND Report*, SAND2002–0835, April 2002.

[20] G. Shafer, *A Mathematical Theory of Evidence*. Princeton: Princeton University Press, 1976.

[21] P. Smets, R. Kennes, "The transferable belief model," *Artificial Intelligence*, vol. 66, pp. 191–234, 1994.

[22] T. J. Stewart, "A critical survey on the status of multiple criteria decision making theory and practice," *OMEGA International Journal of Management Science*, vol. 20, pp. 569–586, 1992.

[23] J. Wang, J. B. Yang, P. Sen, "Multi-person and multi-attribute design evaluations using evidential reasoning based on subjective safety and cost analysis," *Reliability Engineering and System Safety*, vol. 52, pp. 113–127, 1996.

[24] R. R. Yager, "On the Dempster-Shafer framework and new combination rules," *Information Sciences*, vol. 41, pp. 93–137, 1987.

[25] R. R. Yager, "Non-monotonic reasoning with belief structures," in *Advances in the Dempster-Shafer Theory of Evidence*, R. R. Yager, M. Fedrizzi, J. Kacprzyk, Eds., New York: Wiley, 1994, pp. 534–554.

[26] R. R. Yager, M. Fedrizzi, J. Kacprzyk, Eds., *Advances in the Dempster-Shafer Theory of Evidence*. New York: Wiley, 1994.

[27] R. R. Yager, "Decision-making under various types of uncertainties," *Journal of Intelligent* & *Fuzzy Systems*, vol. 3, pp. 317–323, 1995.

[28] J. B. Yang, M. G. Singh, "An evidential reasoning approach for multiple attribute decision making with uncertainty," *IEEE Transactions on Systems, Man, and Cybernetics*, vol. 24, pp. 1–18, 1994.

[29] J. B. Yang, P. Sen, "A general multi-level evaluation process for hyprid multiple attribute decision making with uncertainty," *IEEE Transactions on Systems, Man, and Cybernetics*, vol. 24, pp. 1458–1473, 1994.

[30] J. B. Yang, P. Sen, "Multiple attribute design evaluation of large engineering products using the evidential reasoning approach," *Journal of Engineering Design*, vol. 8, pp. 211–230, 1997.

[31] J. B. Yang, "Rule and utility based evidential reasoning approach for multiple attribute decision analysis under uncertainty," *European Journal of Operational Research*, vol. 131, no. 1, pp. 31–61, 2001.

[32] J. B. Yang, D. L. Xu, "Nonlinear information aggregation via evidential reasoning in multiattribute decision analysis under uncertainty," *IEEE Transactions on Systems, Man, and Cybernetics*, vol. 32, pp. 376–393, 2002.

[33] J. B. Yang, D. L. Xu, "On the evidential reasoning algorithm for multiple attribute decision analysis under uncertainty," *IEEE Transactions on Systems, Man, and Cybernetics*, vol. 32, pp. 289–304, 2002.

[34] L. A. Zadeh, "Reviews of Books: A Methematical Theory of Evidence", *The AI Magazine*, vol. 5, pp. 81–83, 1984.

[35] Z. J. Zhang, J. B. Yang, D. L. Xu, "A hierarchical analysis model for multiobjective decision making," in *Analysis, Design and Evaluation of Man-Machine Systems*, selected papers from the 4th IFAC/IFIP/IFORS/IEA Conference, Xian, P.R. China, September 1989. Pergamon, Oxford, UK, 1990, pp. 13–18.