Leftmost Outermost Revisited*

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Abstract

We present an elementary proof of the classical result that the leftmost outermost strategy is normalizing for left-normal orthogonal rewrite systems. Our proof is local and extends to hyper-normalization and weakly orthogonal systems. Based on the new proof, we study basic normalization, i.e., we study normalization if the set of considered starting terms is restricted to basic terms. This allows us to weaken the left-normality restriction. We show that the leftmost outermost strategy is hyper-normalizing for basically left-normal orthogonal rewrite systems. This shift of focus greatly extends the applicability of the classical result, as evidenced by the experimental data provided.

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1 Introduction

The (hyper-)normalization of the leftmost outermost strategy is a fundamental result in Combinatory Logic and λ-calculus. The importance of hyper-normalization as opposed to normalization stems from the fact that this property is essential to show that all partial recursive functions are definable in λ-calculus and Combinatory Logic. Consequently, numerous (hyper-)normalization proofs can be found in the literature and the result is of foundational interest.

On the other hand, as already observed by O'Donnell [20] effective normalization results are of significant practical interest. Functional programming languages need to be efficiently implemented. For this it is mandatory to study computable and normalizing strategies. Our motivation is mainly concerned with such practical considerations. Consider the term rewrite system (TRS for short) consisting of the rewrite rules

$$\begin{aligned}
\mathsf{primes}(n) &\to \mathsf{take}(n, \mathsf{sieve}(\mathsf{from}(\mathsf{s}(\mathsf{s}(0))))) & \mathsf{filter}(0, y : z, w) &\to 0 : \mathsf{filter}(w, z, w)\\
\mathsf{sieve}(0 : y) &\to \mathsf{sieve}(y) & \mathsf{filter}(\mathsf{s}(x), y : z, w) &\to y : \mathsf{filter}(x, z, w)\\
\mathsf{sieve}(\mathsf{s}(x) : y) &\to \mathsf{s}(x) : \mathsf{sieve}(\mathsf{filter}(x, y, x)) & \mathsf{take}(0, y) &\to \mathsf{nil}\\
\mathsf{from}(x) &\to x : \mathsf{from}(\mathsf{s}(x)) & \mathsf{take}(\mathsf{s}(n), x : y) &\to x : \mathsf{take}(n, y)
\end{aligned}$$

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encoding the sieve of Eratosthenes.

When normalizing a term like primes(s(s(s(0)))), it is important to adopt a good evaluation strategy in order to avoid getting trapped in an infinite computation, which happens for instance with any innermost strategy. Efficiency is another desirable property of a strategy.

Since the TRS is orthogonal, we know that the maximal (parallel) outermost strategy is normalizing (see O'Donnell [20]) but it is also known that the maximal outermost strategy is not optimal in the sense that redexes are contracted which do not contribute to the normal form. Needed reduction is an optimal one-step normalizing strategy for orthogonal TRSs but in general not computable [15]. As a matter of fact, our example TRS happens to be strongly sequential and hence admits a computable optimal one-step strategy [16]. This strategy can be implemented using advanced data structures (matching dags [16], definitional trees [2] for constructor TRSs). Moreover, showing strong sequentiality of orthogonal TRSs is a non-trivial matter [7].

What about a popular and easy to implement strategy like leftmost outermost? Since left-normality is not satisfied, we actually do not know whether the leftmost outermost strategy is normalizing for this TRS. In this situation, we propose a shift of focus. Instead of contemplating normalization of *all* terms, we restrict our attention to *specific* starting terms, following the example of the term primes(s(s(s(0)))). That is, for practical considerations it seems sufficient if we restrict the set of starting terms to basic terms, which are terms that contain exactly one defined symbol, at the root position. This allows us to replace left-normality by a significant weaker restriction. This restriction, which we name *basic left-normality*, is satisfied for the above and many other non-left-normal TRSs, as witnessed by the experimental data that we present in this paper.

The proof is based on *usable replacement maps*, which were originally introduced by Fernández [10] for innermost termination analysis and adapted for complexity analysis of (full) rewriting in [14]. Effective computation of an approximation of usable replacement maps based on unification and fixed point computation is established in [14]. Employing this approximation in the context of basic left-normality yields an easily decidable criterion that ensures the normalization of the leftmost outermost strategy. Furthermore the strategy itself is very easy to implement.

There is a strong and ongoing trend to certify well-established results in all areas of rewriting. Certification not only helps to identify bugs in automated tools, but may also reveal mistakes in the underlying theory. Moreover, the huge body of research in this area shows that it is not unrealistic to aim for the certification of competitive tools. As our motivation is practical, certifiability of strategy tools is clearly of interest to us. Thus in addition to providing a simple and easy to implement strategy for basically left-normal TRS, we provide a formal foundation that is eventually machine-checkable, in order to yield certified evaluations.

Contribution. In this paper we introduce the class of basically left-normal weakly orthogonal TRSs, for which we establish the fundamental result of hyper-normalization for the leftmost outermost strategy starting from basic terms. Along the way we present an elementary proof of the hyper-normalization of the leftmost outermost strategy for the class of left-normal weakly orthogonal TRSs, which is a known result (Toyama [25]). Our proof is based on abstract quasi-commutation properties in connection with a careful analysis of the interplay of single leftmost outermost steps and parallel non-leftmost outermost steps. This gives rise to a local proof which lends itself better to future formalization efforts. We provide experimental evidence which clearly shows the applicability of our result.

Organization. The remainder of the paper is organized as follows. In the next section we recall some rewriting preliminaries and we present the abstract results on which our hyper-normalization proof is based. Section 3 contains the new proof that the leftmost outermost strategy is hyper-normalizing for left-normal weakly orthogonal TRSs. This result is extended to basic hyper-normalization in Section 4, where we also report on our experiments. In Section 5, we discuss related work. Finally, in Section 6, we conclude, where we also discuss future work.

2 Preliminaries

We assume familiarity with term rewriting and all that (e.g., [3]) and only shortly recall notions that are used in the following.

An overlap $(\ell_1 \to r_1, p, \ell_2 \to r_2)_\mu$ of a TRS $\mathcal{R}$ consists of variants $\ell_1 \to r_1$ and $\ell_2 \to r_2$ of rules of $\mathcal{R}$ without common variables, a position $p \in \mathcal{P}\mathsf{os}_\mathcal{F}(\ell_2)$, and a most general unifier μ of ℓ_1 and $\ell_2|_p$. If $p = \epsilon$ then we require that $\ell_1 \to r_1$ and $\ell_2 \to r_2$ are not variants of each other. The pair $((\ell_2\mu)[r_1\mu]_p, r_2\mu)$ is called a critical pair of $\mathcal{R}$. A pair (s,t) is trivial if $s = t$. Left-linear TRSs without critical pairs are called *orthogonal*. A left-linear TRS is *weakly orthogonal* if all critical pairs are trivial.

▶ **Definition 1.** The relations $<_\mathsf{l}$ and $<_\mathsf{lo}$ on positions is inductively defined as follows:

$$\frac{i < j}{i\,p <_\mathsf{l} j\,q} \qquad \frac{p <_\mathsf{l} q}{i\,p <_\mathsf{l} i\,q} \qquad \frac{p \neq \epsilon}{\epsilon <_\mathsf{lo} p} \qquad \frac{i < j}{i\,p <_\mathsf{lo} j\,q} \qquad \frac{p <_\mathsf{lo} q}{i\,p <_\mathsf{lo} i\,q}$$

Here i and j are positive integers and p and q are positions. Distinct positions are called *parallel* if they are comparable with $<_\mathsf{l}$. For a set of parallel positions Q we write $p <_\mathsf{lo} Q$ if $p <_\mathsf{lo} q$ for all $q \in Q$.

We have $p <_\mathsf{l} q$ if and only if position p is to the left of q. It is easy to see that $<_\mathsf{lo}$ is a total order on positions. It is the union of $<_\mathsf{l}$ and the standard prefix order $<$. So $p <_\mathsf{lo} q$ if and only if p is to the left of q or p is strictly above q.

▶ **Definition 2.** The rewrite step that contracts a redex at a position p is denoted by $\to_p$. We write $\mathsf{lo}(t)$ for the smallest redex position with respect to $<_\mathsf{lo}$. A redex at position $\mathsf{lo}(t)$ is called *leftmost-outermost* and we write $t \xrightarrow{\mathsf{lo}} u$ for $t \to_{\mathsf{lo}(t)} u$.

▶ **Definition 3.** A term t is called *left-normal* if function symbols precede variables when t is written in prefix notation. Formally, if $p \in \mathcal{P}\mathsf{os}_\mathcal{V}(t)$ and $q \in \mathcal{P}\mathsf{os}(t)$ with $p <_\mathsf{lo} q$ then $q \in \mathcal{P}\mathsf{os}_\mathcal{V}(t)$. A TRS $\mathcal{R}$ is *left-normal* if the left-hand side ℓ of every rule $\ell \to r \in \mathcal{R}$ is a left-normal term.

▶ **Definition 4.** Parallel rewriting is inductively defined by the following three clauses:

- $t \mathrel{+\!\!\!\!\!\to}_\varnothing t$ for every term t,
- $\ell\sigma \mathrel{+\!\!\!\!\!\to}_{\{\epsilon\}} r\sigma$ for every rewrite rule $\ell \to r$ and substitution σ,
- $f(s_1,\dots,s_n) \mathrel{+\!\!\!\!\!\to}_P f(t_1,\dots,t_n)$ if $s_i \mathrel{+\!\!\!\!\!\to}_{P_i} t_i$ for all $1 \leqslant i \leqslant n$ and $P = \{i\,p \mid 1 \leqslant i \leqslant n$ and $p \in P_i\}$.

We write $s \mathrel{+\!\!\!\!\!\to} t$ if $s \mathrel{+\!\!\!\!\!\to}_P t$ for some set of positions P.

We conclude this section by stating the abstract results that will be used in Section 3. We deal with abstract rewrite systems (ARSs) $\langle A, \to\rangle$ whose relation $\to$ is decomposed into two, not necessarily disjoint, parts $\to_\alpha$ and $\to_\beta$. In order to reduce the number of arrows, we denote the individual relations $\to_\alpha$ and $\to_\beta$ of such an ARS $\langle A, \{\to_\alpha, \to_\beta\}\rangle$ simply by α and β.

Let $\mathcal{A} = \langle A, \{\alpha, \beta\}\rangle$ be an ARS. We say that α *quasi-commutes* over β if the inclusion $\to_\beta \cdot \to_\alpha \subseteq \to_\alpha \cdot \to^*$ holds. The following easy result was first stated in [12].

▶ **Lemma 5.** *Let $\mathcal{A} = \langle A, \{\alpha, \beta\}\rangle$ be an ARS. If $\to_\beta \cdot \to_\alpha \subseteq \to_\alpha^* \cdot \to_\beta^=$ then $\to^* \subseteq \to_\alpha^* \cdot \to_\beta^*$.*

Proof. From the assumption we obtain $\to_\beta \cdot \to_\alpha^* \subseteq \to_\alpha^* \cdot \to_\beta^=$ by a straightforward induction proof. We show $\to^n \subseteq \to_\alpha^* \cdot \to_\beta^*$ by induction on $n \geqslant 0$. If $n = 0$ the inclusion holds trivially. Suppose $a \to b \to^n c$. The induction hypothesis yields $a \to b \to_\alpha^* \cdot \to_\beta^* c$. We distinguish two cases. If $a \to_\alpha b$ then $a \to_\alpha^* \cdot \to_\beta^* c$ holds without further ado. If $a \to_\beta b$ then we obtain $a \to_\alpha^* \cdot \to_\beta^= \cdot \to_\beta^* c$ from the strengthened assumption and thus also $a \to_\alpha^* \cdot \to_\beta^* c$. ◀

The following result is due to Bachmair and Dershowitz [4]. Here α/β denotes the relation $\to_\beta^* \cdot \to_\alpha \cdot \to_\beta^*$ and α/β-termination is perhaps better known as the termination of α relative to β.

▶ **Lemma 6.** *Let $\mathcal{A} = \langle A, \{\alpha, \beta\}\rangle$ be an ARS. If α quasi-commutes over β then every α-terminating element is α/β-terminating.*

Proof. From the quasi-commutation assumption we obtain $\to_\beta^* \cdot \to_\alpha \subseteq \to_\alpha \cdot \to^*$ by a straightforward induction argument. So α quasi-commutes over β^*. We prove that every α-terminating element $a \in A$ is α/β-terminating by well-founded induction on the restriction of $\to_\alpha$ to α-terminating elements, which is a well-founded relation. If $a \in \mathsf{NF}(\alpha/\beta)$ then the claim is trivial. Consider an arbitrary step $a \to_{\alpha/\beta} b$, i.e., $a \to_\beta^* \cdot \to_\alpha \cdot \to_\beta^* b$. Using the quasi-commutation of α over β, the latter sequence can be written as $a \to_\alpha a' \to^* b$. The element a' is α-terminating because a is α-terminating and $a \to_\alpha a'$. Hence the induction hypothesis yields that a' is α/β-terminating. Since $\to^* = \to_\beta^* \cup \to_{\alpha/\beta}^*$ and α/β-terminating elements are preserved under $\to_\beta$, it follows that b is α/β-terminating. Because this holds for any step $a \to_{\alpha/\beta} b$, element a is α/β-terminating. ◀

A rewrite strategy $\mathcal{S}$ for an ARS $\mathcal{A} = \langle A, \to_\mathcal{A}\rangle$ is a relation $\to_\mathcal{S}$ such that $\to_\mathcal{S} \subseteq \to_\mathcal{A}^+$ and $\mathsf{NF}(\to_\mathcal{S}) = \mathsf{NF}(\mathcal{A})$. A *one-step* strategy $\mathcal{S}$ satisfies $\to_\mathcal{S} \subseteq \to_\mathcal{A}$. We say that a strategy $\mathcal{S}$ is *deterministic* if $a = b$ whenever $a \;{}_\mathcal{S}\!\leftarrow \cdot \to_\mathcal{S} b$. A rewrite strategy $\mathcal{S}$ for an ARS $\mathcal{A}$ is *normalizing* if every $\mathcal{A}$-normalizing element is $\mathcal{S}$-terminating. Here an element a is $\mathcal{A}$-normalizing if $a \to_\mathcal{A}^* b$ for some $b \in \mathsf{NF}(\mathcal{A})$. We call $\mathcal{S}$ *hyper-normalizing* if every $\mathcal{A}$-normalizing element is $\mathcal{S}/\mathcal{A}$-terminating. Normalization is the property that by repeatedly performing steps according the strategy a normal form will be computed, provided the starting term has a normal form. Hyper-normalization is a much stronger property. It guarantees that normal forms will still be computed even if between successive strategy steps arbitrary but finitely many other steps are performed.

▶ **Lemma 7.** *A deterministic rewrite strategy $\mathcal{S}$ for an ARS $\mathcal{A}$ is normalizing if there is an ARS $\mathcal{B}$ with $\to_\mathcal{A}^* \subseteq \to_\mathcal{S}^* \cdot \to_\mathcal{B}^*$ and $\mathsf{NF}(\mathcal{A}) \subseteq \mathsf{NF}(\mathcal{B}^{-1})$.*

Proof. If $a \to_\mathcal{A}^! b$ then $a \to_\mathcal{S}^* c \to_\mathcal{B}^* b$ for some c. Because $b \in \mathsf{NF}(\mathcal{A}) \subseteq \mathsf{NF}(\mathcal{B}^{-1})$, $c = b$ and thus $a \to_\mathcal{S}^! b$. Since $\mathcal{S}$ is deterministic, it is terminating on a. ◀

▶ **Theorem 8.** *A normalizing rewrite strategy $\mathcal{S}$ for an ARS $\mathcal{A}$ is hyper-normalizing if $\mathcal{S}$ quasi-commutes over $\mathcal{A}$.*

Proof. Since every $\mathcal{S}$-terminating element is $\mathcal{S}/\mathcal{A}$-terminating according to Lemma 6, the result follows from the definitions of normalization and hyper-normalization. ◀

3 Left-Normal Weakly Orthogonal Rewrite Systems

In this section we present a simple proof of the hyper-normalization of the leftmost outermost strategy for the class of left-normal weakly orthogonal TRSs. The result is well-known and different proofs can be found in the literature, especially for Combinatory Logic and left-normal orthogonal TRSs, cf. the remarks on related work in Section 5. We give full proof details in order to ease future certification efforts. Let $\mathcal{R}$ be a TRS and let $\to$ denote the induced rewrite relation.

▶ **Definition 9.** A position p in t *overlaps with* q (from above) if there are a rule $\ell \to r \in \mathcal{R}$ and $p' \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell)$ such that $t|_p$ is an instance of ℓ and $q = pp'$.

The key to our proof is the following restriction of parallel rewriting.

▶ **Definition 10.** We write $s \overset{\neg\mathsf{lo}}{\mathrel{-\!\!\!\|\!\!\!\rightarrow}} t$ if $s \mathrel{-\!\!\!\|\!\!\!\rightarrow}_P t$ such that $\mathsf{lo}(s)$ overlaps with none of the positions in P. If P is a singleton set, we may write $s \xrightarrow{\neg\mathsf{lo}} t$.

For *orthogonal* TRSs we have $s \overset{\neg\mathsf{lo}}{\mathrel{-\!\!\!\|\!\!\!\rightarrow}} t$ if and only if $s \mathrel{-\!\!\!\|\!\!\!\rightarrow}_P t$ with $\mathsf{lo}(s) \notin P$. We start our analysis with a number of results that do not rely on left-normality.

The following lemma is obvious from the definition of weak orthogonality, and will be used silently throughout the remainder of this section.

▶ **Lemma 11.** *If $s \to_p t$, $s \to_q u$, and p overlaps with q then $t = u$.*

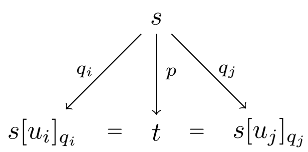
▶ **Lemma 12.** *The identity $\to\ = \xrightarrow{\mathsf{lo}} \cup \xrightarrow{\neg\mathsf{lo}}$ holds for every weakly orthogonal TRS.*

Proof. The inclusion from right to left is obvious. Suppose $s \to_p t$. If $p = \mathsf{lo}(s)$ then $s \xrightarrow{\mathsf{lo}} t$ by definition. If $\mathsf{lo}(s)$ overlaps with p then $s \to_{\mathsf{lo}(s)} t$ by weak orthogonality and thus also $s \xrightarrow{\mathsf{lo}} t$. In the remaining case we have $s \xrightarrow{\neg\mathsf{lo}} t$ by definition. ◀

The next result is essentially due to Takahashi [22].

▶ **Lemma 13.** *Suppose $s \to_p t$ and $s \mathrel{-\!\!\!\|\!\!\!\rightarrow}_Q u$ such that p overlaps with all positions in Q. If $|Q| \geqslant 2$ then $s = t = u$.*

Proof. Let $Q = \{q_1, \ldots, q_n\}$ with $n \geqslant 2$. For $1 \leqslant i \leqslant n$ we denote the subterm of u at position q_i by u_i. So $s \to_{q_i} s[u_i]_{q_i}$. Now for $i \neq j$ we obtain



Since q_i and q_j are parallel, we have $u_i = (s[u_i]_{q_i})|_{q_i} = (s[u_j]_{q_j})|_{q_i} = s|_{q_i}$. Consequently $s \to_{q_i} s[u_i]_{q_i} = s$ and thus $s = t$. Since this holds for all $1 \leqslant i \leqslant n$ we obtain $u = s$. ◀

▶ **Lemma 14.** *The inclusion $\mathrel{-\!\!\!\|\!\!\!\rightarrow}\ \subseteq\ \xrightarrow{\mathsf{lo}}{}^{*} \cdot \overset{\neg\mathsf{lo}}{\mathrel{-\!\!\!\|\!\!\!\rightarrow}}$ holds for every weakly orthogonal TRS.*

Proof. Suppose $s \mathrel{-\!\!\!\|\!\!\!\rightarrow}_P t$. We use induction on $|P|$. Let $P_1 = \{p \in P \mid \mathsf{lo}(s) \text{ overlaps with } p\}$ and $P_2 = P \setminus P_1$. There exists a term u such that $s \mathrel{-\!\!\!\|\!\!\!\rightarrow}_{P_1} u \mathrel{-\!\!\!\|\!\!\!\rightarrow}_{P_2} t$. If $P_1 = \varnothing$ then $s = u$ and if $|P_1| \geqslant 2$ then $s = u$ follows from Lemma 13, and thus we have $s \overset{\neg\mathsf{lo}}{\mathrel{-\!\!\!\|\!\!\!\rightarrow}}_{P_2} t$. If $|P_1| = 1$ then we obtain $s \xrightarrow{\mathsf{lo}} u$ from weak orthogonality. The induction hypothesis yields $u \xrightarrow{\mathsf{lo}}{}^{*} \cdot \overset{\neg\mathsf{lo}}{\mathrel{-\!\!\!\|\!\!\!\rightarrow}} t$ and thus also $s \xrightarrow{\mathsf{lo}}{}^{*} \cdot \overset{\neg\mathsf{lo}}{\mathrel{-\!\!\!\|\!\!\!\rightarrow}} t$. ◀

▶ **Lemma 15.** *The leftmost outermost strategy is deterministic for every weakly orthogonal TRS.*

Proof. Since the leftmost outermost redex position is unique in any reducible term, the statement holds for any TRS without non-trivial overlays. In particular, it holds for every weakly orthogonal TRS. ◀

Because of the inclusion $\twoheadrightarrow\!\!\!\!\!\!+ \;\subseteq\; \to^*$, which holds for arbitrary TRSs, every parallel rewrite step can be serialized into a sequence of rewrite steps. For orthogonal TRSs it can be shown that every $\xrightarrow{\neg\mathsf{lo}}\!\!\!\!\!\!+\;$ step can be serialized into a sequence of $\xrightarrow{\neg\mathsf{lo}}$ steps. However, the following (original) example shows that serialization of $\xrightarrow{\neg\mathsf{lo}}\!\!\!\!\!\!+\;$ does not extend to weakly orthogonal TRSs.

▶ **Example 16.** Consider the weakly-orthogonal TRS $\mathcal{R}$ consisting of the four rewrite rules

$$\mathsf{a} \to \mathsf{b} \qquad \mathsf{f}(\mathsf{g}(\mathsf{a},\mathsf{b})) \to \mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b}))$$
$$\mathsf{g}(x,y) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{a})) \to \mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b}))$$

Note that $\mathcal{R}$ is left-normal. Let $s = \mathsf{f}(\mathsf{g}(\mathsf{a},\mathsf{a}))$ and $t = \mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b}))$. We have $\mathsf{lo}(s) = 1$ and $s \xrightarrow{\neg\mathsf{lo}}\!\!\!\!\!\!+\; t$ since $s \twoheadrightarrow\!\!\!\!\!\!+_{\{11,12\}}\; t$ and position 1 does not overlap with 11 or 12 in s. From s we can perform two different $\xrightarrow{\neg\mathsf{lo}}$ steps:

$$s \xrightarrow{\neg\mathsf{lo}} \mathsf{f}(\mathsf{g}(\mathsf{a},\mathsf{b})) \qquad s \xrightarrow{\neg\mathsf{lo}} \mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{a}))$$

In both cases we obtain a term which is in normal form with respect to $\xrightarrow{\neg\mathsf{lo}}$ because the created redex at the root position equals a left-hand side.

For the results that follow we need the restriction to *left-normal* weakly orthogonal TRSs.

▶ **Definition 17.** A position p is said to be *below* a term t if $p \geqslant q$ for some $q \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(t)$.

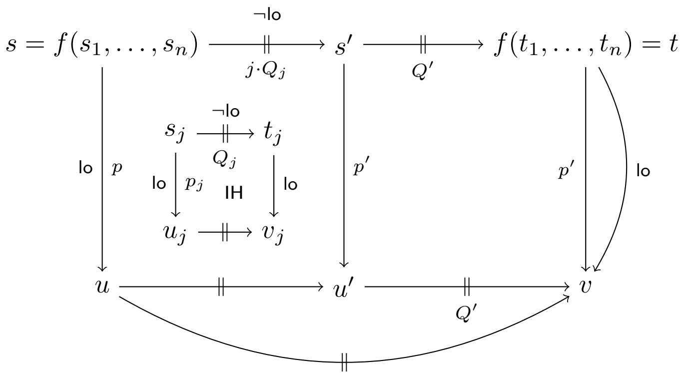
▶ **Lemma 18.** *Let t be a left-normal term and σ a substitution. If $p, q \in \mathcal{P}\mathsf{os}(t\sigma)$ such that p is below t and $p <_{\mathsf{lo}} q$ then q is below t.*

Proof. We have $p \geqslant q'$ for some position $q' \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(t)$. If $q \geqslant q'$ then q is below t. If $q \geqslant q'$ does not hold then we must have $p <_{\mathsf{l}} q$ and $q' <_{\mathsf{l}} q$. If $q \in \mathcal{P}\mathsf{os}(t)$ then $q \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(t)$ by left-normality. Otherwise $q > q''$ for some position $q'' \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(t)$. In both cases we conclude that q is below t. ◀

The next result can be viewed as a special case of the Parallel Moves Lemma. Although the statement appears plausible, the proof is subtle because of weak orthogonality. Below, we denote $i \cdot P$ by $\{i\,p \mid p \in P\}$, and write $\sigma \twoheadrightarrow\!\!\!\!\!\!+\; \tau\ [X]$ if $x\sigma \twoheadrightarrow\!\!\!\!\!\!+\; x\tau$ holds for all $x \in X$.

▶ **Lemma 19.** *If $\mathcal{R}$ is a left-normal weakly orthogonal TRS then $\xleftarrow{\mathsf{lo}} \cdot \xrightarrow{\neg\mathsf{lo}}\!\!\!\!\!\!+\; \subseteq\; \twoheadrightarrow\!\!\!\!\!\!+\; \cdot \xleftarrow{\mathsf{lo}}$.*

Proof. Let $s \xrightarrow{\neg\mathsf{lo}}\!\!\!\!\!\!+_Q\; t$ and $s \xrightarrow{\mathsf{lo}}_p u$. By induction on the sum $\|Q\|$ of the lengths of the positions in Q we show the existence of a term v such that $t \xrightarrow{\mathsf{lo}} v$ and $u \twoheadrightarrow\!\!\!\!\!\!+\; v$. If $Q = \varnothing$ then $s = t$ and we simply take $u = v$. If $\epsilon \in Q$ then $Q = \{\epsilon\}$ and so there is nothing to show since the assumption $s \xrightarrow{\neg\mathsf{lo}}\!\!\!\!\!\!+_Q\; t$ is violated. In the remaining case we have $s = f(s_1,\ldots,s_n)$, $t = f(t_1,\ldots,t_n)$, $s_i \twoheadrightarrow\!\!\!\!\!\!+_{Q_i}\; t_i$ for all $1 \leqslant i \leqslant n$, and $Q = \{i\,q \mid 1 \leqslant i \leqslant n \text{ and } q \in Q_i\}$. We distinguish two further cases, depending on the position p.



$$
\begin{array}{llll}
p = j \cdot p_j & p <_{\mathsf{lo}} j \cdot Q_j & p' = j \cdot \mathsf{lo}(t_j) \quad p' <_{\mathsf{l}} Q' & p' <_{\mathsf{lo}} j \cdot Q_j \\
s' = f(s_1, \ldots, s_{j-1}, t_j, s_{j+1}, \ldots, s_n) & & u' = f(s_1, \ldots, s_{j-1}, v_j, s_{j+1}, \ldots, s_n) & \\
u = f(s_1, \ldots, s_{j-1}, u_j, s_{j+1}, \ldots, s_n) & & v = f(s_1, \ldots, s_{j-1}, v_j, t_{j+1}, \ldots, t_n) &
\end{array}
$$

Figure 1 The critical case in the proof of Lemma 19.

- If $p = \epsilon$ then there exist a rewrite rule $\ell \to r$ and a substitution σ such that $s = \ell\sigma$ and $u = r\sigma$. Since the root symbol of s is f, we may write $\ell = f(\ell_1, \ldots, \ell_n)$. Fix $i \in \{1, \ldots, n\}$. We have $s_i = \ell_i\sigma \mathrel{+\!\!\!\!\!\!\longrightarrow} t_i$. Since $\mathsf{lo}(s) = \epsilon$ does not overlap with positions in Q, all steps in $s \overset{\neg\mathsf{lo}}{\mathrel{+\!\!\!\!\!\!\longrightarrow}}_Q t$ take place in the substitution σ. Using the linearity of ℓ, it is not difficult to prove the existence of a substitution τ_i such that $t_i = \ell_i\tau_i$ and $\sigma \mathrel{+\!\!\!\!\!\!\longrightarrow} \tau_i\ [\mathcal{V}\mathsf{ar}(\ell_i)]$. We assume without loss of generality that $\mathsf{dom}(\tau_i) \subseteq \mathcal{V}\mathsf{ar}(\ell_i)$. Otherwise, we can always consider the restriction of τ_i to $\mathcal{V}\mathsf{ar}(\ell_i)$ due to the linearity of ℓ. Thus the substitution $\tau = \tau_1 \cup \cdots \cup \tau_n$ is well-defined and satisfies $\ell\tau = t$ and $\sigma \mathrel{+\!\!\!\!\!\!\longrightarrow} \tau\ [\mathcal{V}\mathsf{ar}(\ell)]$. Let $v = r\tau$. Since parallel rewriting is closed under substitutions, we obtain $u = r\sigma \mathrel{+\!\!\!\!\!\!\longrightarrow} r\tau = v$. Furthermore, we obviously have $t = \ell\tau \to_p r\tau = v$ with $\mathsf{lo}(t) = \epsilon = p$.
- If $p \neq \epsilon$ then $p = j\,p_j$ for some $1 \leqslant j \leqslant n$ and position $p_j \in \mathcal{P}\mathsf{os}(s_j)$. This case is illustrated in Figure 1. We have $\|Q_j\| < \|Q\|$ and $Q_i = \varnothing$ for all $1 \leqslant i < j$ because $p <_{\mathsf{lo}} Q$. Moreover, $s_j \overset{\neg\mathsf{lo}}{\mathrel{+\!\!\!\!\!\!\longrightarrow}}_{Q_j} t_j$ follows from $s \overset{\neg\mathsf{lo}}{\mathrel{+\!\!\!\!\!\!\longrightarrow}}_Q t$. Hence we can apply the induction hypothesis, yielding a term v_j such that $t_j \xrightarrow{\mathsf{lo}} v_j$ and $u_j \mathrel{+\!\!\!\!\!\!\longrightarrow} v_j$. Let $s' = s[t_j]_j$, $u' = s[v_j]_j$, and $p' = j \cdot \mathsf{lo}(t_j)$. We have $u \mathrel{+\!\!\!\!\!\!\longrightarrow} u'$, $s' \to_{p'} u'$, and $s' \mathrel{+\!\!\!\!\!\!\longrightarrow}_{Q'} t$ with $Q' = Q \setminus Q_j = \{i\,q \mid j < i \leqslant n \text{ and } q \in Q_i\}$. Obviously, $p' <_{\mathsf{l}} Q'$ and thus there exists a term v such that $u' \mathrel{+\!\!\!\!\!\!\longrightarrow}_{Q'} v$ and $t \to_{p'} v$. The parallel steps $u \mathrel{+\!\!\!\!\!\!\longrightarrow} u'$ and $u' \mathrel{+\!\!\!\!\!\!\longrightarrow}_{Q'} v$ can be combined into a single parallel step $u \mathrel{+\!\!\!\!\!\!\longrightarrow} v$ because the redexes contracted in $u \mathrel{+\!\!\!\!\!\!\longrightarrow} u'$ are below position j and thus to the left of all positions in Q'. It remains to show that $t \xrightarrow{\mathsf{lo}} v$. This is obvious if $p' = \mathsf{lo}(t)$. So suppose $p' \neq \mathsf{lo}(t)$, which implies $\mathsf{lo}(t) = \epsilon$. So $t = \ell\sigma$ for some rewrite rule $\ell = f(\ell_1, \ldots, \ell_n) \to r$ and substitution σ. We distinguish two further cases.
  - If p' is not below ℓ then we obtain $t \xrightarrow{\mathsf{lo}} v$ from weak orthogonality.
  - If p' is below ℓ then, since $p' <_{\mathsf{l}} Q'$, all positions in Q' are below ℓ according to Lemma 18. Since

    $$p' = j \cdot \mathsf{lo}(t_j) \leqslant j \cdot p_j = p \leqslant_{\mathsf{lo}} j \cdot Q_j$$

    the same holds for the positions in $j \cdot Q_j$. It follows that s is an instance of ℓ, contradicting $\mathsf{lo}(s) \neq \epsilon$. ◀

The following example shows the necessity of left-normality in Lemma 19.

▶ **Example 20.** Consider the orthogonal TRS consisting of the rewrite rules

$$\mathsf{a} \to \mathsf{b} \qquad\qquad \mathsf{f}(x, \mathsf{b}) \to \mathsf{c}$$

and the term $s = \mathsf{f}(\mathsf{a}, \mathsf{a})$. We have $s \xrightarrow{\mathsf{lo}} \mathsf{f}(\mathsf{b}, \mathsf{a})$ and $s \xrightarrow{\neg\mathsf{lo}} \mathsf{f}(\mathsf{a}, \mathsf{b})$ but there is no term t such that $\mathsf{f}(\mathsf{b}, \mathsf{a}) \mathrel{-\!\!\|\!\!\to} t$ and $\mathsf{f}(\mathsf{a}, \mathsf{b}) \xrightarrow{\mathsf{lo}} t$.

▶ **Lemma 21.** *The inclusion* $\overset{\neg\mathsf{lo}}{\mathrel{-\!\!\|\!\!\to}} \subseteq \xrightarrow{\mathsf{lo}} \cdot \mathrel{-\!\!\|\!\!\to} \cdot \xleftarrow{\mathsf{lo}} \cup =$ *holds for every left-normal weakly orthogonal TRS.*

Proof. Let $s \overset{\neg\mathsf{lo}}{\mathrel{-\!\!\|\!\!\to}}_P t$. If $P = \varnothing$ then $s = t$. If $P \neq \varnothing$ then s is reducible and thus $s \xrightarrow{\mathsf{lo}} u$ for some term u. We obtain $u \mathrel{-\!\!\|\!\!\to} \cdot \xleftarrow{\mathsf{lo}} t$ from Lemma 19 and thus $s \xrightarrow{\mathsf{lo}} \cdot \mathrel{-\!\!\|\!\!\to} \cdot \xleftarrow{\mathsf{lo}} t$ as desired. ◀

Combining Lemmata 14, 15 and 21 gives the following result.

▶ **Corollary 22.** *The inclusion* $\overset{\neg\mathsf{lo}}{\mathrel{-\!\!\|\!\!\to}} \cdot \xrightarrow{\mathsf{lo}} \subseteq \xrightarrow{\mathsf{lo}}{}^{+} \cdot \overset{\neg\mathsf{lo}}{\mathrel{-\!\!\|\!\!\to}}$ *holds for every left-normal weakly orthogonal TRS.*

Proof. We have

$$\begin{aligned}
\overset{\neg\mathsf{lo}}{\mathrel{-\!\!\|\!\!\to}} \cdot \xrightarrow{\mathsf{lo}} &\subseteq (\xrightarrow{\mathsf{lo}} \cdot \mathrel{-\!\!\|\!\!\to} \cdot \xleftarrow{\mathsf{lo}} \cup =) \cdot \xrightarrow{\mathsf{lo}} && \text{(Lemma 21)}\\
&= \xrightarrow{\mathsf{lo}} \cdot \mathrel{-\!\!\|\!\!\to} \cdot \xleftarrow{\mathsf{lo}} \cdot \xrightarrow{\mathsf{lo}} \cup = \cdot \xrightarrow{\mathsf{lo}} &&\\
&\subseteq \xrightarrow{\mathsf{lo}} \cdot \mathrel{-\!\!\|\!\!\to} \cdot = \cup \xrightarrow{\mathsf{lo}} && \text{(Lemma 15)}\\
&= \xrightarrow{\mathsf{lo}} \cdot \mathrel{-\!\!\|\!\!\to} &&\\
&\subseteq \xrightarrow{\mathsf{lo}} \cdot \xrightarrow{\mathsf{lo}}{}^{*} \cdot \overset{\neg\mathsf{lo}}{\mathrel{-\!\!\|\!\!\to}} && \text{(Lemma 14)}\\
&= \xrightarrow{\mathsf{lo}}{}^{+} \cdot \overset{\neg\mathsf{lo}}{\mathrel{-\!\!\|\!\!\to}} &&
\end{aligned}$$ ◀

▶ **Corollary 23.** *The relation* $\xrightarrow{\mathsf{lo}}$ *quasi-commutes over* $\overset{\neg\mathsf{lo}}{\mathrel{-\!\!\|\!\!\to}}$ *for every left-normal weakly orthogonal TRS.*

Proof. This follows from the preceding corollary and the inclusion $\xrightarrow{\mathsf{lo}}{}^{+} \cdot \overset{\neg\mathsf{lo}}{\mathrel{-\!\!\|\!\!\to}} \subseteq \xrightarrow{\mathsf{lo}} \cdot \to^{*}$. ◀

▶ **Corollary 24.** *The inclusion* $\to^{*} \subseteq \xrightarrow{\mathsf{lo}}{}^{*} \cdot \overset{\neg\mathsf{lo}}{\mathrel{-\!\!\|\!\!\to}}{}^{*}$ *holds for every left-normal weakly orthogonal TRS.*

Proof. This follows from Corollary 22 in connection with Lemmata 12 and 5 (with $\to_\alpha = \xrightarrow{\mathsf{lo}}$ and $\to_\beta = \overset{\neg\mathsf{lo}}{\mathrel{-\!\!\|\!\!\to}}$). ◀

We arrive at the (hyper-)normalization theorem.

▶ **Theorem 25.** *The leftmost outermost strategy is hyper-normalizing for every left-normal weakly orthogonal TRS.*

Proof. Normalization of $\xrightarrow{\mathsf{lo}}$ is obtained from Lemma 7, Corollary 24, and the inclusion $\mathsf{NF}(\mathcal{R}) \subseteq \mathsf{NF}(\xleftarrow{\neg\mathsf{lo}})$ which follows from Lemma 19: If $t \notin \mathsf{NF}(\xleftarrow{\neg\mathsf{lo}})$ then $s \xrightarrow{\neg\mathsf{lo}} t$ for some term s and thus $s \xrightarrow{\mathsf{lo}} u$ for some term u and hence $u \mathrel{-\!\!\|\!\!\to} \cdot \xleftarrow{\mathsf{lo}} t$, so $t \notin \mathsf{NF}(\mathcal{R})$. By combining the normalization of $\xrightarrow{\mathsf{lo}}$ with Theorem 8 and Corollary 23, hyper-normalization of $\xrightarrow{\mathsf{lo}}$ is concluded. ◀

4 Basic Normalization

We recall a few notions from context-sensitive rewriting.

▶ **Definition 26.** A *replacement map* associates every n-ary function symbol to a subset of $\{1,\ldots,n\}$. Let μ be a replacement map. The set $\mathcal{P}\mathsf{os}^\mu(t)$ of *active* positions in t is inductively defined as follows:

$$\mathcal{P}\mathsf{os}^\mu(t) = \begin{cases} \{\epsilon\} & \text{if } t \text{ is a variable} \\ \{\epsilon\} \cup \{ip \mid i \in \mu(f),\ 1 \leqslant i \leqslant n, \text{ and } p \in \mathcal{P}\mathsf{os}^\mu(t_i)\} & \text{if } t = f(t_1,\ldots,t_n) \end{cases}$$

The set $\mathcal{P}\mathsf{os}_\mathcal{V}(t) \cap \mathcal{P}\mathsf{os}^\mu(t)$ is abbreviated to $\mathcal{P}\mathsf{os}^\mu_\mathcal{V}(t)$.

We introduce basic normalization. Let $\mathcal{R}$ be a TRS and $\mathcal{D} = \{\mathsf{root}(\ell) \mid \ell \to r \in \mathcal{R}\}$ be the set of all defined symbols in $\mathcal{R}$. Terms t with $\mathcal{P}\mathsf{os}_\mathcal{D}(t) = \{\epsilon\}$ are called *basic terms.*

▶ **Definition 27.** A rewrite strategy $\mathcal{S}$ for a TRS $\mathcal{R}$ is *basically normalizing* if every $\mathcal{R}$-normalizing basic term is $\mathcal{S}$-terminating.

By recasting the basic term condition of basic normalization in a replacement map for context-sensitive rewriting, we establish a powerful criterion for basic normalization. As a basis of this formulation, we use usable replacement maps [10, 14].

▶ **Definition 28.** A term t is *accessible* if $s \to^* t$ for some basic term s. A replacement map μ for a TRS $\mathcal{R}$ is *usable* if all redex positions in every accessible term t are included in $\mathcal{P}\mathsf{os}^\mu(t)$.

An effective technique for computation of usable replacement maps based on unification and fixed point computation is suggested in [14]. The method adapts the cap-function ICAP suitably [11]. The key observation is that through the cap-function usable arguments can be delineated which is formalizable as a monotone operator $\Upsilon^\mathcal{R}$. Applying Tarski's fixed point theorem we conclude the existence of a least fixed point which yields the desired approximation, cf. [14, Definition 9].

We define basically left-normal TRSs. The notion of μ-left-normality in the following definition was introduced in [18] for normalization of context-sensitive rewriting.

▶ **Definition 29.** A term t is called left-normal with respect to a replacement map μ, or simply *μ-left-normal,* if $p \in \mathcal{P}\mathsf{os}^\mu_\mathcal{V}(t)$ and $q \in \mathcal{P}\mathsf{os}^\mu(t)$ with $p <_{\mathsf{lo}} q$ imply $q \in \mathcal{P}\mathsf{os}_\mathcal{V}(t)$. Let $\mathcal{R}$ be a TRS with a usable replacement map μ. The TRS $\mathcal{R}$ is *basically left-normal* if the left-hand side ℓ of every rule $\ell \to r \in \mathcal{R}$ is μ-left-normal.

▶ **Example 30.** Consider the orthogonal TRS in the introduction. The map μ given by

$$\mu(\mathsf{sieve}) = \{1\} \qquad \mu(\mathsf{primes}) = \mu(\mathsf{from}) = \mu(\mathsf{s}) = \mu(\mathsf{0}) = \varnothing$$
$$\mu(\mathsf{filter}) = \mu(\mathsf{take}) = \mu(:) = \{2\}$$

is a usable replacement map. For example, consider $t = \mathsf{take}(\mathsf{s}(n), x : y)$. There are no active positions $p \in \mathcal{P}\mathsf{os}^\mu_\mathcal{V}(t) = \{22\}$ and $q \in \mathcal{P}\mathsf{os}^\mu(t) = \{\epsilon, 2, 22\}$ such that $p <_{\mathsf{lo}} q$. So t is μ-left-normal. In a similar way one can verify basic left-normality of all other left-hand sides of the TRS left-normal. Hence, the TRS is basic left-normal with respect to μ.

We show that the leftmost outermost strategy is basically normalizing for basically left-normal TRSs.

	left-normal	basically left-normal
# of TRSs	75	150
total time (in seconds)	0.63	1.03

Table 1 Experimental results on 161 weakly orthogonal TRSs.

▶ **Lemma 31.** *Let μ be a replacement map, t a μ-left-normal term, and σ a substitution. If $p, q \in \mathcal{P}\mathsf{os}^{\mu}(t\sigma)$ such that p is below t and $p <_{\mathsf{lo}} q$ then q is below t.*

Proof. The proof of Lemma 18 goes through after replacing $\mathcal{P}\mathsf{os}_{\mathcal{V}}(t)$ with $\mathcal{P}\mathsf{os}^{\mu}_{\mathcal{V}}(t)$: We have $p \geqslant q'$ for some position $q' \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(t)$. Hence $q' \in \mathcal{P}\mathsf{os}^{\mu}_{\mathcal{V}}(t)$ follows from $p \in \mathcal{P}\mathsf{os}^{\mu}(t\sigma)$. If $q \geqslant q'$ then q is below t. If $q \geqslant q'$ does not hold then we must have $p <_{\mathsf{l}} q$ and $q' <_{\mathsf{l}} q$ for otherwise the assumption $p <_{\mathsf{lo}} q$ would be violated. If $q \in \mathcal{P}\mathsf{os}^{\mu}(t)$ then $q \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(t)$ by μ-left-normality. Otherwise $q > q''$ for some position $q'' \in \mathcal{P}\mathsf{os}^{\mu}_{\mathcal{V}}(t) \subseteq \mathcal{P}\mathsf{os}_{\mathcal{V}}(t)$. In both cases we conclude that q is below t. ◀

Let $\rightsquigarrow$ be a relation on terms. The *accessible version* $\rightsquigarrow_{\mathsf{a}}$ is defined as follows: $s \rightsquigarrow_{\mathsf{a}} t$ if s is an accessible term and $s \rightsquigarrow t$. In Section 3 we used left-normality to show Lemma 19. The next lemma is its counterpart for basic hyper-normalization.

▶ **Lemma 32.** *Let $\mathcal{R}$ be a weakly orthogonal TRS with a usable replacement map μ. If $\mathcal{R}$ is basically left-normal then the inclusion ${}_{\mathsf{a}}\xleftarrow{\text{lo}} \cdot \overset{\neg\mathsf{lo}}{+\!\!\!+\!\!\!\!\!\twoheadrightarrow}_{\mathsf{a}} \subseteq +\!\!\!+\!\!\!\!\!\twoheadrightarrow_{\mathsf{a}} \cdot {}_{\mathsf{a}}\xleftarrow{\text{lo}}$ holds.*

Proof. Let $s \overset{\neg\mathsf{lo}}{+\!\!\!+\!\!\!\!\!\rightarrow}_{\mathsf{a}} t$ and $s \xrightarrow{\mathsf{lo}}_{\mathsf{a}} u$. Since s is accessible and μ is usable, $s \overset{\neg\mathsf{lo}}{+\!\!\!+\!\!\!\!\!\rightarrow}_{Q} t$ and $s \xrightarrow{\mathsf{lo}}_{p} u$ hold for some $Q \subseteq \mathcal{P}\mathsf{os}^{\mu}(s)$ and $p \in \mathcal{P}\mathsf{os}^{\mu}(s)$. We obtain $t +\!\!\!+\!\!\!\!\!\rightarrow \cdot \xleftarrow{\mathsf{lo}} u$ as in the proof of Lemma 19, provided that Lemma 31 is used instead of Lemma 18. Since accessibility of s carries over to t and u, we conclude $t +\!\!\!+\!\!\!\!\!\twoheadrightarrow_{\mathsf{a}} \cdot {}_{\mathsf{a}}\xleftarrow{\text{lo}} u$. ◀

By using Lemma 32, one can lift all statements (and proofs) in Lemma 21 and Corollaries 22, 23, and 24 to the accessible version in a straightforward way.

▶ **Theorem 33.** *The leftmost outermost strategy is basically hyper-normalizing for every basically left-normal weakly orthogonal TRS.* ◀

We implemented and tested Theorems 8 and 33 on a collection of 161 weakly orthogonal TRSs, consisting of the 153 systems for innermost, outermost, and context-sensitive rewriting in version 8.0.7 of the Termination Problem Data Base (TPDB)[1] and 9 systems from van de Pol's examples for strategy annotations [21].[2] Usable replacement maps for Theorem 33 are estimated by the fixed point computation of [14, Definition 9]. Table 1 summarizes the results.[3]

5 Related Work

The (hyper-)normalization of the leftmost outermost strategy is a fundamental result in Combinatory Logic and λ-calculus. The importance of hyper-normalization as opposed to

[1] `http://termcomp.uibk.ac.at/`
[2] `http://wwwhome.cs.utwente.nl/~vdpol/jitty/`
[3] Details are available at: `http://www.jaist.ac.jp/~hirokawa/15bn/`

normalization stems from the fact that this property is essential to show that all partial recursive functions are definable in λ-calculus and Combinatory Logic. Consequently, numerous (hyper-)normalization proofs can be found in the literature. Below we comment on some of them.

Combinatory Logic and λ-calculus. The usual argument that the leftmost outermost strategy is (hyper-)normalizing in Combinatory Logic or λ-calculus employs the standardization theorem [8, 5], which in itself is based on the study of residuals. Avoiding residuals, Kashima presents in [17] an interesting inductive treatment of standardization and thus provides a simple proof of hyper-normalization of the leftmost outermost strategy for the λ-calculus.

It is perhaps worthy of note that the proof of the (hyper-)normalization theorem is absent from the well-known textbook by Hindley and Seldin [13]. The other recent book covering Combinatory Logic by Bimbó [6, Lemma 2.2.8] contains the following short and *incomplete* proof of the normalization theorem (here $\rhd_1$ refers to the rewrite relation of Combinatory Logic, "lmrs" stands for the leftmost (outermost) strategy, and Theorem 2.1.14 is the Church-Rosser theorem):

> *If a CL-term has an nf, then there is a $\rhd_1$ reduction sequence of finite length. If the leftmost reduction sequence is a finite $\rhd_1$ reduction sequence, then by theorem 2.1.14, the last term is the nf of the starting term.*
> *The other possibility is that the lmrs gives us an infinite $\rhd_1$ reduction sequence. Therefore, if some redex other than the leftmost one is reduced in another $\rhd_1$ reduction sequence, then the leftmost redex remains in the term as long as the lmrs is not followed, i.e., the term will not reduce to its nf.* *(qed)*

Left-normal (orthogonal) term rewrite systems. O'Donnell [20] introduced left-normality and proved the normalization of the leftmost outermost strategy for left-normal orthogonal TRSs. A modern account of his proof, which is based on residual theory and cofinality of leftmost-fair reductions, can be found in [23, Section 4.9].

Hyper-normalization of the leftmost outermost strategy for left-normal orthogonal TRSs is obtained by van Oostrom and de Vrijer in [23, Theorem 9.3.21] as a corollary of the more general statement that the leftmost outermost strategy is a *needed* strategy for left-normal orthogonal TRSs.

Extending upon [24] Toyama proves in [25] that external reduction, which is a variation of needed reduction, is a normalizing strategy for the class of left-linear root balanced joinable external TRSs. Note that the studied TRSs may be ambiguous. As a corollary he obtains the hyper-normalization of the leftmost outermost strategy for the class of left-linear left-normal root balanced joinable TRSs. The latter class includes all left-normal weakly orthogonal TRSs. It is an open problem whether our proof method extends to left-linear left-normal root balanced joinable TRSs.

All these proofs can be characterized as global in the sense that definitions refer to properties of rewrite sequences rather than single (parallel) steps, which are manipulated throughout the proof. We believe this will hamper formalization efforts. In contrast, our proof is elementary and local, as it makes essential use of abstract commutation properties.

Commutation properties In [1] Accattoli introduces an abstract framework for factorization, relying on commutation properties similar to those exploited in Lemma 5 and 6 above. The framework is general in the sense that it applies to a multitude of explicit substitutions calculi. However, its applicability in our context is less straightforward. In order to employ a *square factorization system* [1, Definition 3.3] to prove our Corollary 24, we would need to decompose the relations $\xrightarrow{\mathsf{lo}}$ and $\xrightarrow{\neg\mathsf{lo}}\!\!\!\!\!\!\!\!+\;\;$ into four relations ($\xrightarrow{\mathsf{lo}}_1$, $\xrightarrow{\mathsf{lo}}_2$, $\xrightarrow{\neg\mathsf{lo}}_1$, $\xrightarrow{\neg\mathsf{lo}}_2$), such that $\xrightarrow{\mathsf{lo}}_1$ and $\xrightarrow{\neg\mathsf{lo}}_1$ are terminating; a non-trivial task.

Dershowitz argues in [9, Note 20] that quasi-commutation applies to Combinatory Logic and orthogonal TRSs, in connection with the inclusion $\xleftarrow{\mathsf{lo}} \cdot \xrightarrow{\neg\mathsf{lo}} \subseteq \xrightarrow{\neg\mathsf{lo}}^* \cdot \xleftarrow{\mathsf{lo}}$. Apart from the missing left-normality condition, the inclusion does not hold, not even for Combinatory Logic as $\mathsf{I}x \xleftarrow{\mathsf{lo}} \mathsf{II}x \xrightarrow{\neg\mathsf{lo}} \mathsf{I}x$ but not $\mathsf{I}x \xrightarrow{\neg\mathsf{lo}}^* \cdot \xleftarrow{\mathsf{lo}} \mathsf{I}x$.

6 Conclusion

In this paper we have presented an elementary proof of the classical result that the leftmost outermost strategy is normalizing for left-normal orthogonal rewrite systems. Our proof is local and extends to hyper-normalization and weakly orthogonal systems. Our interest in leftmost outermost stems from the observation that archetypical TRSs often fail the definition of left-normality, while morally these TRSs are left-normal, if the set of starting terms is suitable restricted.

Based on this observation, we introduced basic normalization, i.e., normalization if the set of considered starting terms is restricted to basic terms. This allowed us to weaken the left-normality restriction. Building upon our new proof, we have shown that the leftmost outermost strategy is hyper-normalizing for basically left-normal weakly orthogonal rewrite systems. This provides a simple and easy to implement strategy for basic terms in a surprisingly large number of cases, as evidenced by the experimental data provided.

Despite the technical challenges found in the generalization of our result to weakly orthogonal systems, we have striven for an elementary proof, which we believe offers itself to future formalization within an interactive theorem prover. Hopefully our results pave the way for future certification efforts in the area of strategies.

In future work we will pursue an experimental and theoretical comparison of our results with the normalizing strategies induced by strongly/inductively [16, 2] sequential TRSs. Furthermore, we seek a simple proof of (hyper-)normalization of maximal (parallel) outermost for weakly orthogonal TRSs (cf. [20, 19]).

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