Non-E-Overlapping, Weakly Shallow, and Non-Collapsing TRSs are Confluent*

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Abstract. A term is *weakly shallow* if each defined function symbol occurs either at the root or in the ground subterms, and a term rewriting system is weakly shallow if both sides of a rewrite rule are weakly shallow. This paper proves that non-E-overlapping, weakly-shallow, and non-collapsing term rewriting systems are confluent by extending *reduction graph* techniques in our previous work [SO10] with *towers of expansions*.

1 Introduction

Confluence of term rewriting systems (TRSs) is undecidable, even for flat TRSs [MOJ06] or length-two string rewrite systems [SW08]. Two decidable subclasses are known: right-linear and shallow TRSs by tree automata techniques [GT05] and terminating TRSs by resolving to finite search [KB70]. Many sufficient conditions have been proposed, and they are classified into two categories.

- Local confluence for terminating TRSs [KB70]. It was extended to TRSs with relative termination [HM11,KH12]. Another criterion comes with the decomposition to linear and terminating non-linear TRSs [LDJ14]. It requires conditions for the existence of well-founded *ranking*.
- Peak elimination with an explicit well-founded measure. Lots of works explore left-linear TRSs under the non-overlapping condition and its extensions [Ros73,Hue80,Toy87,Oos95,Oku98,OO97]. For non-linear TRSs, there are quite few works [TO95,GOO98] under the non-E-overlapping condition (which coincides with non-overlapping if left-linear) and additional restrictions that allow to define such measures.

We have proposed a different methodology, called a *reduction graph* [SO10], and shown that "*weakly non-overlapping, shallow, and non-collapsing TRSs are confluent*". An original idea comes from observing that, when non-E-overlapping,

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peak-elimination uses only "*copies*" of reductions in an original rewrite sequences. Thus, if we focus on terms appearing in peak elimination, they are finitely many. We regard a rewrite relation over these terms as a directed graph, and construct a confluent directed acyclic graph (DAG) in a bottom-up manner, in which the shallowness assumption works. The keys are, such a DAG always has a unique normal form (if it is finite), and convergence is preserved if we add an arbitrary reduction starting from a normal form. Our reduction graph technique is carefully designed to preserve both acyclicity and finiteness.

This paper introduces the notion of *towers of expansions*, which extends a reduction graph by adding terms and edges expanded with function symbols in an on-demand way, and shows that "*weakly shallow, non-E-overlapping, and non-collapsing TRSs are confluent*". A term is weakly shallow if each defined function symbol appears either at the root or in the ground subterms, and a TRS is weakly shallow if the both sides of rules are weakly shallow. It is worth mentioning:

- A Turing machine is simulated by a weakly shallow TRS [Klo93] (see Remark 1), and many decision problems, such as the word problem, termination and confluence, are undecidable [MOM12]. Note that the word problem is decidable for shallow TRSs [CHJ94]. The fact distinguishes these classes.
- The non-E-overlapping property is undecidable for weakly shallow TRSs [MOM12]. A decidable sufficient condition is *strongly non-overlapping*, where a TRS is *strongly non-overlapping* if its linearization is non-overlapping [OO89]. Here, these conditions are the same when left-linear.
- Our result gives a new criterion for confluence provers of TRSs. For instance,

  $$\{d(x,x) \to h(x), f(x) \to d(x, f(c)), c \to f(c), h(x) \to h(g(x))\}$$

  is shown to be confluent only by ours.

Remark 1. Let Q, Σ and Γ ($\supseteq \Sigma$) be finite sets of states, input symbols and tape symbols of a Turing machine M, respectively. Let $\delta :\ Q \times \Gamma \to Q \times \Gamma \times \{\text{left}, \text{right}\}$ be the transition function of M. Each configuration $a_1 \cdots a_i q a_{i+1} \cdots a_n \in \Gamma^+ Q \Gamma^+$ (where $q \in Q$) is represented by a term $q(a_i \cdots a_1(\$), a_{i+1} \cdots a_n(\$))$ where arities of function symbols q, a_j ($1 \leq j \leq n$) and \$ are 2, 1 and 0, respectively. The corresponding TRS R_M consists of rewriting rules below:

$$\begin{array}{ll} q(x, a(y)) \to p(b(x), y) & \text{if } \delta(q,a) = (p, b, \text{right}), \\ q(a'(x), a(y)) \to p(x, a'(b(y))) & \text{if } \delta(q,a) = (p, b, \text{left}) \end{array}$$

2 Preliminaries

2.1 Abstract Reduction System

For a binary relation $\to$, we use $\leftarrow$, $\leftrightarrow$, $\to^+$ and $\to^*$ for the inverse relation, the symmetric closure, the transitive closure, and the reflexive and transitive closure of $\to$, respectively. We use $\cdot$ for the composition operation of two relations.

An *abstract reduction system* (ARS) is a directed graph $G = \langle V, \to \rangle$ with reduction $\to \subseteq V \times V$. If $(u, v) \in \to$, we write it as $u \to v$. An element u of V is *($\to$-)normal* if there exists no $v \in V$ with $u \to v$. We sometimes call a normal element a *normal form*. For subsets V' and V'' of V, $\to|_{V' \times V''} = \to \cap (V' \times V'')$.

Let $G = \langle V, \to \rangle$ be an ARS. We say G is *finite* if V is finite, *confluent* if $\leftarrow^* \cdot \to^* \subseteq \to^* \cdot \leftarrow^*$, *Church-Rosser* (*CR*) if $\leftrightarrow^* \subseteq \to^* \cdot \leftarrow^*$, and *terminating* if it does not admit an infinite reduction sequence from a term. G is *convergent* if it is confluent and terminating. Note that confluence and CR are equivalent.

We refer standard terminology in graphs. Let $G = \langle V, \to \rangle$ and $G' = \langle V', \to' \rangle$ be ARSs. We use $V_{G'}$ and $\to_{G'}$ to denote V' and $\to'$, respectively. An edge $v \to u$ is an *outgoing-edge* of v and an *incoming-edge* of u, and v is the *initial vertex* of $\to$. A vertex v is $\to$-normal if it has no outgoing-edges. The union of graphs is defined as $G \cup G' = \langle V \cup V', \to \cup \to' \rangle$. We say

- G is *connected* if $(u, v) \in \leftrightarrow^*$ for each $u, v \in V$.
- G' *includes* G, denoted by $G' \supseteq G$, if $V' \supseteq V$ and $\to' \supseteq \to$.
- G' *weakly subsumes* G, denoted by $G' \sqsupseteq G$, if $V' \supseteq V$ and $\leftrightarrow'^* \supseteq \to$.
- G' *conservatively extends* G, if $V' \supseteq V$ and $\leftrightarrow'^*|_{V \times V} = \leftrightarrow^*$.

The weak subsumption relation $\sqsupseteq$ is transitive.

2.2 Term Rewriting System

Let F be a finite set of function symbols, and X be an enumerable set of variables with $F \cap X = \emptyset$. $\mathrm{T}(F, X)$ denotes the set of terms constructed from F and X and $\mathrm{Var}(t)$ denotes the set of variables occurring in a term t. A *ground* term is a term in $\mathrm{T}(F, \emptyset)$. The set of positions in t is $\mathrm{Pos}(t)$, and the *root* position is ε. For $p \in \mathrm{Pos}(t)$, the subterm of t at position p is denoted by $t|_p$. The root symbol of t is $\mathrm{root}(t)$, and the set of positions in t whose symbols are in S is denoted by $\mathrm{Pos}_S(t) = \{p \mid \mathrm{root}(t|_p) \in S\}$. The term obtained from t by replacing its subterm at position p with s is denoted by $t[s]_p$. The *size* $|t|$ of a term t is $|\mathrm{Pos}(t)|$. As notational convention, we use s, t, u, v, w for terms, x, y for variables, a, b, c, f, g for function symbols, p, q for positions, and σ, θ for substitutions.

We define $\mathrm{sub}(t)$ as $\mathrm{sub}(x) = \emptyset$ and $\mathrm{sub}(t) = \{t_1, \ldots, t_n\}$ if $t = f(t_1, \ldots, t_n)$. A *rewrite rule* is a pair (ℓ, r) of terms such that $\ell \notin X$ and $\mathrm{Var}(\ell) \supseteq \mathrm{Var}(r)$. We write it $\ell \to r$. A *term rewriting system* (TRS) is a finite set R of rewrite rules. The *rewrite relation* of R on $\mathrm{T}(F, X)$ is denoted by $\underset{R}{\to}$. We sometimes write $s \overset{p}{\underset{R}{\to}} t$ to indicate the *rewrite step* at the position p. Let $s \overset{p}{\underset{R}{\to}} t$. It is a *top reduction* if $p = \varepsilon$. Otherwise, it is an *inner reduction*, written as $s \overset{\varepsilon<}{\underset{R}{\to}} t$.

Given a TRS R, the set D of *defined symbols* is $\{\mathrm{root}(\ell) \mid \ell \to r \in R\}$. The set C of *constructor symbols* is $F \setminus D$. For $T \subseteq \mathrm{T}(F, X)$ and $f \in F$, we use $T|_f$ to denote $\{s \in T \mid \mathrm{root}(s) = f\}$. For a subset F' of F, we use $T|_{F'}$ to denote the union $\cup_{f \in F'} T|_f$.

A *constructor term* is a term in $\mathrm{T}(C, X)$, and a *semi-constructor term* is a term in which defined function symbols appear only in the ground subterms. A term is *shallow* if the length $|p|$ is 0 or 1 for every position p of variables in the term. A *weakly shallow term* is a term in which defined function symbols appear only either at the root or in the ground subterms (i.e., $p \neq \varepsilon$ and $\mathrm{root}(s|_p) \in D$ imply that $s|_p$ is ground). Note that every shallow term is weakly shallow.

A rewrite rule $\ell \to r$ is *weakly shallow* if ℓ and r are weakly shallow, and *collapsing* if r is a variable. A TRS is *weakly shallow* if each rewrite rule is weakly shallow. A TRS is *non-collapsing* if it contains no collapsing rules.

Example 2. A TRS R_1 is weakly shallow and non-collapsing.

$$R_1 = \{f(x,x) \to a,\ f(x,g(x)) \to b,\ c \to g(c)\} \text{ [Hue80]}$$

Let $\ell_1 \to r_1$ and $\ell_2 \to r_2$ be rewrite rules in a TRS R. Let p be a position in ℓ_1 such that $\ell_1|_p$ is not a variable. If there exist substitutions θ_1, θ_2 such that $\ell_1|_p\theta_1 = \ell_2\theta_2$ (resp. $\ell_1|_p\theta_1 \overset{\varepsilon<}{\underset{R}{\leftrightarrow}}{}^* \ell_2\theta_2$), we say that the two rules are *overlapping* (resp. *E-overlapping*), except that $p = \varepsilon$ and the two rules are identical (up to renaming variables). A TRS R is *overlapping* (resp. *E-overlapping*) if it contains a pair of overlapping (resp. E-overlapping) rules. Note that TRS R_1 in Example 2 is E-overlapping since $f(c,c) \overset{\varepsilon<}{\underset{R}{\leftrightarrow}}{}^* f(c,g(c))$.

3 Extensions of Convergent Abstract Reduction Systems

This section describes a transformation system from a finite ARS to obtain a convergent (i.e., terminating and confluent) ARS that preserves the connectivity.

Let $G = \langle V, \to\rangle$ be an ARS. If G is finite and convergent, then we use a function $\downarrow_G$ (called the choice mapping) that takes an element of V and returns the normal form [SO10]. We also use $v{\downarrow_G}$ instead of $\downarrow_G(v)$.

Definition 3. *For ARSs $G_1 = \langle V_1, \to_1\rangle$ and $G_2 = \langle V_2, \to_2\rangle$, we say that $G_1 \cup G_2$ is the* hierarchical combination *of G_2 with G_1, denoted by $G_1 \gtrdot G_2$, if $\to_1 \subseteq (V_1 \setminus V_2) \times V_1$.*

Proposition 4. *$G_1 \gtrdot G_2$ is terminating if both G_1 and G_1 are so.*

Lemma 5. *Let $G_1 \gtrdot G_2$ be a confluent and hierarchical combination of ARSs. If a confluent ARS G_3 weakly subsumes G_2 and $G_1 \gtrdot G_3$ is a hierarchical combination, then $G_1 \gtrdot G_3$ is confluent.*

Proof. We use $\langle V_i, \to_i\rangle$ to denote G_i. Let $\alpha : u' \leftarrow^*_{G_1 \gtrdot G_3} u \to^*_{G_1 \gtrdot G_3} u''$. If $u \in V_3$, only $\to_3$ appears in α, and hence $u' \to^*_3 \cdot \leftarrow^*_3 u''$ follows from the confluence of G_3. Otherwise, α is represented as $u' \leftarrow^*_3 v' \leftarrow^*_1 u \to^*_1 v'' \to^*_3 u''$. Since $v' \to^*_1 w' \to^*_2 \cdot \leftarrow^*_2 w'' \leftarrow^*_1 v''$ for some w' and w'' (from the confluence of $G_1 \gtrdot G_2$) and $G_2 \sqsubseteq G_3$, we obtain $u' \leftarrow^*_3 v' \to^*_1 w' \leftrightarrow^*_3 w'' \leftarrow^*_1 v'' \to^*_3 u''$. Since $G_1 \gtrdot G_3$ is a hierarchical combination, $v' = w'$ if $v' \in V_3$, and $v' = u'$ otherwise. Hence, $u' \to^*_1 \cdot \leftrightarrow^*_3 w'$. Similarly either $v'' = w''$ or $v'' = u''$. Thus, $u' \to^*_1 \cdot \leftrightarrow^*_3 \cdot \leftarrow^*_1 u''$. The confluence of G_3 gives $u' \to^*_1 \cdot \to^*_3 \cdot \leftarrow^*_3 \cdot \leftarrow^*_1 u''$, and $u' \to^*_{G_1 \gtrdot G_3} \cdot \leftarrow^*_{G_1 \gtrdot G_3} u''$. ◻

In the sequel, we generalize properties of ARSs obtained in [SO10].

Definition 6. *Let $G = \langle V, \to \rangle$ be a convergent ARS. Let v, v' be vertices such that $v \neq v'$ and if $v \in V$ then v is $\to$-normal. Then G', denoted by $G \multimap (v \to v')$, is defined as follows (see Fig. 1):*

$$\begin{cases} \langle V \cup \{v'\}, \to \cup \{(v,v')\}\rangle & \text{if } v \in V \text{ and } v' \notin V \quad (1) \\ \langle V, \to \cup \{(v,v')\}\rangle & \text{if } v, v' \in V \text{ and } v' \not\leftrightarrow^* v \quad (2) \\ \langle V, \to \setminus \{(v',v'') \mid v' \to v''\} \cup \{(v,v')\}\rangle & \text{if } v, v' \in V \text{ and } v' \leftrightarrow^* v \quad (3) \\ \langle V \cup \{v, v'\}, \to \cup \{(v,v')\}\rangle & \text{if } v \notin V \quad (4) \end{cases}$$

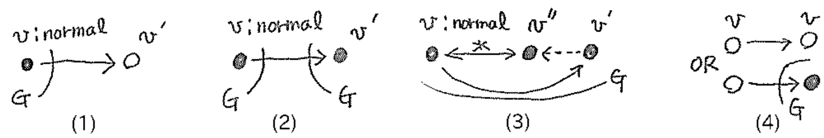

Fig. 1. Adding an edge to a convergent ARS

Note that v' becomes a normal form of G' when the first or the third transformation is applied.

Proposition 7. *For a convergent ARS G, the ARS $G' = G \multimap (v \to v')$ is convergent, and satisfies $G' \sqsupseteq G$.*

We represent $G \multimap (v_0 \to v_1) \multimap (v_1 \to v_2) \multimap \cdots \multimap (v_{n-1} \to v_n)$ as $G \multimap (v_0 \to v_1 \to \cdots \to v_n)$ (if Definition 6 can be repeatedly applied).

Proposition 8. *Let $G = \langle V, \to \rangle$ be a convergent ARS. Let $v_0, v_1, \ldots, v_n$ satisfy $v_i \neq v_j$ (for $i \neq j$), and one of the following conditions:*

(1) $v_0 \in V$, v_0 is $\to$-normal, and $v_i \in V$ implies $v_i \leftrightarrow^* v_0$ for each $i (< n)$,
(2) $v_0, \cdots, v_{n-1} \notin V$.

Then, $G' = G \multimap (v_0 \to v_1 \to \cdots \to v_n)$ is well-defined and convergent, and $G' \sqsupseteq G$ holds.

4 Reduction Graphs

From now on, we fix C and D as the sets of constructors and defined function symbols for a TRS R, respectively. We assume that there exists a constructor with a positive arity in C, otherwise all weakly shallow terms are shallow.

4.1 Reduction Graphs and Monotonic Extension

Definition 9 ([SO10]). *An ARS $G = \langle V, \to \rangle$ is an R-*reduction graph *if V is a finite subset of* $\mathrm{T}(F, X)$ *and $\to \subseteq \underset{R}{\to}$.*

For an R-reduction graph $G = \langle V, \to \rangle$, *inner-edges*, *strict inner-edges*, and *top-edges* are given by $\overset{\varepsilon<}{\to} \; = \; \to \cap \underset{R}{\overset{\varepsilon<}{\to}}$, $\overset{\neq\varepsilon}{\to} \; = \; \to \setminus \underset{R}{\overset{\varepsilon}{\to}}$, and $\overset{\varepsilon}{\to} \; = \; \to \cap \underset{R}{\overset{\varepsilon}{\to}}$, respectively. We use $G^{\varepsilon<}$, $G^{\neq\varepsilon}$, and G^{ε} to denote $\langle V, \overset{\varepsilon<}{\to}\rangle$, $\langle V, \overset{\neq\varepsilon}{\to}\rangle$, and $\langle V, \overset{\varepsilon}{\to}\rangle$, respectively. Remark that for $R = \{a \to b, f(x) \to f(b)\}$ $V = \{f(a), f(b)\}$, and $G = \langle V, \{(f(a), f(b))\}\rangle$, we have $G^{\varepsilon<} = G^{\varepsilon} = G$ and $G^{\neq\varepsilon} = \langle V, \emptyset\rangle$.

For an R-reduction graph $G = \langle V, \to\rangle$ and $F' \subseteq F$, we represent $G|_{F'} = \langle V, \to|_{F'}\rangle$ where $\to|_{F'} = \to|_{V|_{F'} \times V}$. Note that $\to|_C = \to|_{V|_C \times V|_C}$ and $\to = \to|_D \cup \to|_{V|_C \times V|_C}$.

Definition 10. *Let $G = \langle V, \to\rangle$ be an R-reduction graph. The* direct-subterm reduction-graph $\mathrm{sub}(G)$ *of G is $\langle \mathrm{sub}(V), \mathrm{sub}(\to)\rangle$ where*

$$\begin{cases} \mathrm{sub}(V) = \bigcup_{t \in V} \mathrm{sub}(t) \\ \mathrm{sub}(\to) = \{(s_i, t_i) \mid f(s_1, \ldots, s_n) \overset{\varepsilon<}{\to} f(t_1, \ldots, t_n),\ s_i \neq t_i,\ 1 \leq i \leq n\}. \end{cases}$$

An R-reduction graph $G = \langle V, \to\rangle$ is subterm-closed *if* $\mathrm{sub}(G^{\neq\varepsilon}) \sqsubseteq G$.

Lemma 11. *Let $G = \langle V, \to\rangle$ be a subterm-closed R-reduction graph. Assume that (1) $s[t]_p \leftrightarrow^* s[t']_p$, and (2) for any $p' < p$, if $(s[t]_p)|_{p'} \leftrightarrow^* (s[t']_p)|_{p'}$ then $(s[t]_p)|_{p'} \overset{\neq\varepsilon}{\leftrightarrow}{}^* (s[t']_p)|_{p'}$. Then $t \leftrightarrow^* t'$.*

Proof. By induction on $|p|$. If $p = \varepsilon$, trivial. Let $p = iq$ and $s = f(s_1, \ldots, s_n)$. Since $s[t]_p \overset{\neq\varepsilon}{\leftrightarrow}{}^* s[t']_p$ from the assumptions, the subterm-closed property of G implies $s_i[t]_q \leftrightarrow^* s_i[t']_q$. Hence, $t \leftrightarrow^* t'$ holds by induction hypothesis. □

Definition 12. *For a set F' ($\subseteq F$) and an R-reduction graph $G = \langle V, \to\rangle$, the* F'-monotonic extension $M_{F'}(G) = \langle V_1, \to_1\rangle$ *is*

$$\begin{cases} V_1 \ = \{f(s_1, \ldots, s_n) \mid f \in F',\ s_1, \ldots, s_n \in V\}, \\ \to_1 = \{(f(\cdots s \cdots), f(\cdots t \cdots)) \in V_1 \times V_1 \mid s \to t\}. \end{cases}$$

Example 13. As a running example, we use the following TRS, which is non-E-overlapping, non-collapsing, and weakly shallow with $C = \{g\}$ and $D = \{c, f\}$:

$$R_2 = \{f(x, g(x)) \to g^3(x),\ c \to g(c)\ \}.$$

Consider a subterm-closed R_2-reduction graph $G = \langle \{c, g(c), g^2(c)\}, \{(c, g(c))\}\rangle$. In the sequel, we use a simple representation of graphs as $G = \{c \to g(c),\ \ g^2(c)\}$. The C-monotonic extension $M_C(G)$ of G is $M_C(G) = \{g(c) \to g^2(c),\ \ g^3(c)\}$.

Proposition 14. *Let $M_{F'}(G) = \langle V', \to'\rangle$ be the F'-monotonic extension of an R-reduction graph $G = \langle V, \to\rangle$. Then,*

(1) if G is terminating (resp. confluent), then $M_{F'}(G)$ is.
(2) If G is subterm-closed, then for $u, v \in V|_{F'}$, we have (a) $u, v \in V'$, and (b) $u \overset{\neq\varepsilon}{\to} v$ implies $u \leftrightarrow'^* v$.
(3) $\mathrm{sub}(M_{F'}(G)) \subseteq G$ if F' contains a function symbol with a positive arity.

4.2 Constructor Expansion

Definition 15. *For a subterm-closed R-reduction graph G, a* constructor expansion $\overline{M_C}(G)$ *is the hierarchical combination $G|_D \gtrdot M_C(G)$ ($= G|_D \cup M_C(G)$). The k-times application of $\overline{M_C}$ to G is denoted by $\overline{M_C}^k(G)$.*

Example 16. For G in Example 13, the constructor expansions $\overline{M_C}^i(G)$ of G ($i = 1, 3$) are

$$\overline{M_C}(G) = \{c \to g(c) \to g^2(c), \quad g^3(c)\},$$
$$\overline{M_C}^3(G) = \{c \to g(c) \to g^2(c) \to g^3(c) \to g^4(c), \quad g^5(c)\}.$$

Lemma 17. *Let G be a subterm-closed R-reduction graph. Then,*

(1) $\mathrm{sub}(\overline{M_C}(G)^{\neq\varepsilon}) \sqsubseteq G$, *and*
(2) $\to_{G^{\neq\varepsilon}} \subseteq \leftrightarrow^*_{M_F(G)}$, *that is,* $G \sqsubseteq G^\varepsilon \cup M_F(G)$,

Proof. Let $G = \langle V, \to \rangle$. We refer $M_C(G)$ by $G' = \langle V', \to' \rangle$. Thus, for $v \in V'$, $\mathrm{root}(v) \in C$. Note that $\overline{M_C}(G) = G|_D \gtrdot M_C(G) = \langle V' \cup V, \to' \cup \to|_{V|_D \times V} \rangle$.

(1) Due to $\mathrm{sub}(\overline{M_C}(G)^{\neq\varepsilon}) = \mathrm{sub}(G^{\neq\varepsilon}|_D) \cup \mathrm{sub}(M_C(G))$, it is enough to show $\mathrm{sub}(G^{\neq\varepsilon}|_D) \sqsubseteq G$ and $\mathrm{sub}(M_C(G)) \sqsubseteq G$. The former follows from the fact that $\mathrm{sub}(G^{\neq\varepsilon}|_D) \subseteq \mathrm{sub}(G^{\neq\varepsilon})$ and G is subterm-closed. The latter follows from $\mathrm{sub}(M_C(G)) \subseteq G$.
(2) Obvious from Proposition 14 (2). □

Lemma 18. *For a subterm-closed R-reduction graph G,*

(1) $G \sqsubseteq \overline{M_C}(G)$,
(2) $\overline{M_C}(G)$ *is subterm-closed, and*
(3) $\overline{M_C}(G)$ *is convergent if G is convergent.*

Proof. Let $G = \langle V, \to \rangle$. Note that $\overline{M_C}(G) = (G|_D \gtrdot M_C(G)) = \langle V \cup V_{M_C(G)}, \to|_D \cup \to_{M_C(G)} \rangle$.

(1) Since $\to|_{V|_C \times V|_C} \subseteq \overset{\neq\varepsilon}{\to}_G$, we have $\to|_{V|_C \times V|_C} \subseteq \leftrightarrow^*_{M_C(G)}$ (by Proposition 14 (2)), so that $G \sqsubseteq \overline{M_C}(G)$.
(2) By Lemma 17 (1), $\mathrm{sub}(\overline{M_C}(G)^{\neq\varepsilon}) \sqsubseteq G$. Combining this with $G \sqsubseteq \overline{M_C}(G)$, we obtain $\mathrm{sub}(\overline{M_C}(G)^{\neq\varepsilon}) \sqsubseteq \overline{M_C}(G)$. Thus, $\overline{M_C}(G)$ is subterm-closed.
(3) If we show $G' = \langle V|_C, \to|_{V|_C \times V|_C} \rangle \sqsubseteq M_C(G)$, the confluence of $\overline{M_C}(G) = G|_D \gtrdot M_C(G)$ follows from Lemma 5, since $G = G|_D \gtrdot G'$ and $M_C(G)$ is confluent by Proposition 14 (1). Since G is subterm-closed, we have $V|_C \subseteq V_{M_C(G)}$ and $\to|_{V|_C \times V|_C} \subseteq \leftrightarrow^*_{M_C(G)}$ by Proposition 14 (2). Hence, $G' \sqsubseteq M_C(G)$. The termination of $\overline{M_C}(G)$ follows from Proposition 4, since $G|_D$ and $M_C(G)$ are terminating. □

Corollary 19. *For a subterm-closed R-reduction graph G and $k \geq 0$, we have:*

(1) $G \sqsubseteq \overline{M_C}^k(G)$.
(2) $\overline{M_C}^k(G)$ *is subterm-closed.*
(3) $\overline{M_C}^k(G)$ *is convergent, if G is convergent.*

Remark 20. When an R-reduction graph G is subterm-closed, we observe that $\leftrightarrow^*_{\overline{M_C}^k(G)} = \leftrightarrow^*_{G \cup M_C(G) \cup \cdots \cup M_C^k(G)}$ from $\to_{G|_C} \subseteq \leftrightarrow^*_{M_C(G)}$ by Proposition 14 (2).

Proposition 21. *Let G be a subterm-closed R-reduction graph. Then,* $\overline{M_C}^k(G) \sqsubseteq \overline{M_C}^m(G)$ *for $m > k \geq 0$.*

Proof. By $\overline{M_C}^m(G) = \overline{M_C}^{m-k}(\overline{M_C}^k(G))$ and Corollary 19 (1) and (2). □

5 Tower of Constructor Expansions

From now on, let G be a convergent and subterm-closed R-reduction graph. We call $M_F(\overline{M_C}^i(G))$ a *tower of constructor expansions* of G for $i \geq 0$. We use $G_{2_i} = \langle V_{2_i}, \to_{2_i} \rangle$ to denote $M_F(\overline{M_C}^i(G))$.

5.1 Enriching Reduction Graph

We show that there exists a convergent R-reduction graph G_1 with $M_F(G) \sqsubseteq G_1$ such that G_{2_i} is a conservative extension of G_1 for large enough i.

Lemma 22. *For a convergent and subterm-closed R-reduction graph G, there exist k (≥ 0) and an R-reduction graph G_1 satisfying the following conditions.*

i) *G_1 is convergent, and consists of inner-edges.*
ii) *$G_1 \sqsubseteq G_{2_k}$.*
iii) *$u \leftrightarrow^*_{2_i} v$ implies $u \leftrightarrow^*_1 v$ for each $u, v \in V_1$ and i (≥ 0).*
iv) *$M_F(G) \sqsubseteq G_1$.*

Proof. Let $G_1 := M_F(G)$ and $k := 0$. We define a condition iii)' as "iii) holds for all i $(< k)$". Initially, i) holds by Proposition 14 (1) since G is convergent. ii) and iv) hold from $G_1 = M_F(G) = G_{2_0}$, and iii)' holds from $k = 0$.

We transform G_1 so that i), ii), iii)' and iv) are preserved and the number $|V_1/{\leftrightarrow^*_1}|$ of connected components of G_1 decreases. This transformation $(G_1, k) \vdash (G'_1, k')$ continues until iii) eventually holds, since $|V_1/{\leftrightarrow^*_1}|$ is finite.

For current G_1 and k, we assume that i), ii), iii)' and iv) hold. If G_1 fails iii), there exist i with $i \geq k$ and $u, v \in V_1$ such that $u \neq v$ and $(u, v) \in {\leftrightarrow^*_{2_i}} \setminus {\leftrightarrow^*_1}$. We choose such k' as the least i. Remark that G_1 is convergent from i), and $G_{2_{k'}}$ is convergent from Corollary 19 (3) and Proposition 14 (1). Let $\downarrow_1$ and $\downarrow_{2_{k'}}$ be the choice mappings of G_1 and $G_{2_{k'}}$, respectively. Since $G_1 \sqsubseteq G_{2_{k'}}$ from ii) and Proposition 21, we have $(u{\downarrow_1}, v{\downarrow_1}) \in {\leftrightarrow^*_{2_{k'}}}$ and $u{\downarrow_1} \neq v{\downarrow_1}$. From the convergence of $G_{2_{k'}}$, we have

$$\begin{cases} u{\downarrow_1} = u_0 \to_{2_{k'}} u_1 \to_{2_{k'}} \cdots \to_{2_{k'}} u_{n'} \to_{2_{k'}} \cdots \to_{2_{k'}} u_n = (u{\downarrow_1}){\downarrow_{2_{k'}}} \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \shortparallel \qquad\qquad\qquad\qquad\quad \shortparallel \\ v{\downarrow_1} = v_0 \to_{2_{k'}} v_1 \to_{2_{k'}} \cdots \to_{2_{k'}} v_{m'} \to_{2_{k'}} \cdots \to_{2_{k'}} v_m = (v{\downarrow_1}){\downarrow_{2_{k'}}} \end{cases}$$

where (n', m') is the smallest pair under the lexicographic ordering such that $u_{n'} = v_{m'}$. Note that u_j's and v_j's do not necessarily belong to V_1. We define a transformation $(G_1, k) \vdash (G'_1, k')$ with G'_1 to be

$$\begin{cases} G_1 \multimap (u_0 \to \cdots \to u_j) & \text{if there exists (the smallest) } j \text{ such that} \\ & \quad 0 < j \leq n',\ u_j \in V_1, \text{ and } u_j \not\leftrightarrow^*_1 u \\ G_1 \multimap (v_0 \to \cdots \to v_{j'}) & \text{if there exists (the smallest) } j' \text{ such that} \\ & \quad 0 < j' \leq m',\ v_{j'} \in V_1, \text{ and } v_{j'} \not\leftrightarrow^*_1 v \\ G_1 \multimap (u_0 \to \cdots \to u_{n'}) \multimap (v_0 \to \cdots \to v_{m'}) & \text{otherwise.} \end{cases}$$

Since the condition (1) of Proposition 8 holds, i) is preserved. From $G_1 \sqsubseteq G'_1$ iv) holds, and ii) $G'_1 \sqsubseteq G_{2_{k'}}$ by Proposition 21. If $k' = k$, iii)' does not change. If $k' > k$, then $u \leftrightarrow^*_{2_i} v$ implies $u \leftrightarrow^*_1 v$ for i with $k \leq i < k'$, since we chose k' as the least. Hence iii)' holds. In either case, $|V_1/{\leftrightarrow^*_1}|$ decreases. ◻

Example 23. For G in Example 13, Lemma 22 starts from $M_F(G)$, which is displayed by the solid edges in Fig. 2. G_1 is constructed by augmenting the dashed edges with $k = 1$.

$$
\begin{array}{ccccccc}
c & & f(c,c) & \rightarrow & f(g(c),c) & & f(g^2(c),c) \\
 & & \downarrow & & \downarrow & & \downarrow \\
g(c) & & f(c,g(c)) & \rightarrow & f(g(c),g(c)) & \dashrightarrow & f(g^2(c),g(c)) \\
\downarrow & & & & & & \dashdownarrow \\
g^2(c) & & f(c,g^2(c)) & \rightarrow & f(g(c),g^2(c)) & \dashrightarrow & f(g^2(c),g^2(c)) \\
\dashdownarrow & & & & & & \\
g^3(c) & & & & & &
\end{array}
$$

Fig. 2. G_1 constructed by Lemma 22 from G in Example 13

Corollary 24. *Assume that $G_1 = \langle V_1, \rightarrow_1 \rangle$ and h (≥ 0) satisfy the conditions i) to iv) in Lemma 22. Let $v_0, v_1, \ldots, v_n$ satisfy $v_j \neq v_{j'}$ for $j \neq j'$ and v_{j-1} $(\leftrightarrow^*_{2_k} \cap \overset{\varepsilon<}{\underset{R}{\rightarrow}})$ v_j for $1 \leq j \leq n$. If either (1) $v_0 \in V_1$ and v_0 is $\rightarrow_1$-normal, or (2) $v_0, \cdots, v_{n-1} \notin V_1$ and $v_n \in V_1$, then the conditions i) to iv) hold for $G_{1'} = G_1 \multimap (v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_n)$ and $k' = \max(k, h)$.*

Proof. For (1), from iii) of G_1, $v_j \in V_1$ implies $v_j \leftrightarrow^*_1 v_0$. For either case, from i) and iv) of G_1 and Proposition 8, $G_{1'}$ satisfies i) and iv). Since $v_{j-1} \leftrightarrow^*_{2_k} v_j$, $G_{1'}$ immediately satisfies ii). Since $v_0 \in V_1$ or $v_n \in V_1$, $G_{1'}$ satisfies iii). $\square$

5.2 Properties of Tower of Expansions on Weakly Shallow Systems

Lemma 25. *Let R be a non-E-overlapping and weakly shallow TRS. Let $G = \langle V, \rightarrow \rangle$ be a convergent and subterm-closed R-reduction graph, and let $\ell \rightarrow r \in R$.*

(1) If $\ell\sigma \leftrightarrow^*_{2_i} \ell\theta$, then $x\sigma \leftrightarrow^*_{\overline{M_C}^i(G)} x\theta$ for each variable $x \in \mathrm{Var}(\ell)$.
(2) For a weakly shallow term s with $s \notin X$, assume that $x\sigma \leftrightarrow^*_{\overline{M_C}^i(G)} x\theta$ for each variable $x \in \mathrm{Var}(s)$. If $s\sigma \in V_{2_i}$, then $s\sigma \leftrightarrow^*_{2_k} s\theta$ for some k ($\geq i$).
(3) If $\ell\sigma \leftrightarrow^*_{2_i} u$, then there exist a substitution θ and k ($\geq i$) such that u $(\overset{\varepsilon<}{\underset{R}{\rightarrow}} \cap \leftrightarrow^*_{2_k})^*$ $\ell\theta$ and $x\sigma \rightarrow^*_{\overline{M_C}^i(G)} x\theta$ for each variable $x \in \mathrm{Var}(\ell)$.

Proof. Note that G_{2_i} is convergent by Corollary 19 (3) and Proposition 14 (1).

(1) Let $\ell = f(\ell_1, \ldots, \ell_n)$. For each j ($1 \leq j \leq n$), $\ell_j\sigma \leftrightarrow^*_{\overline{M_C}^i(G)} \ell_j\theta$. Since $\overline{M_C}^i(G)$ is convergent by Corollary 19 (3), there exists v_j such that $\ell_j\sigma \rightarrow^*_{\overline{M_C}^i(G)} v_j \leftarrow^*_{\overline{M_C}^i(G)} \ell_j\theta$. Since $\overline{M_C}^i(G)$ is subterm-closed by Corollary 19 (2) and ℓ_j is semi-constructor, we have $x\sigma \leftrightarrow^*_{\overline{M_C}^i(G)} x\theta$ for every $x \in \mathrm{Var}(\ell)$ by Lemma 11.

(2) First, we show that for a semi-constructor term t if $t\sigma \in V_{\overline{M_C}^i(G)}$, there exists k ($\geq i$) such that $t\sigma \leftrightarrow^*_{\overline{M_C}^k(G)} t\theta$ by induction on the structure of t. If t is either a variable or a ground term, immediate. Otherwise, let $t = f(t_1, \ldots, t_n)$ for $f \in C$. Since $\overline{M_C}^i(G)$ is subterm-closed, $t_j\sigma \in V_{\overline{M_C}^i(G)}$ for each j. Hence, induction hypothesis ensures $t_j\sigma \leftrightarrow^*_{\overline{M_C}^{k_j}(G)} t_j\theta$ for some $k_j \geq i$. Since $M_C(\overline{M_C}^i(G)) \subseteq \overline{M_C}^{i+1}(G)$ and Proposition 21, we have $t\sigma \leftrightarrow^*_{\overline{M_C}^k(G)} t\theta$ for $k = 1 + \max\{k_1, \ldots, k_n\}$.
We show the statement (2). Since $s \notin X$, s is represented as $f(s_1, \ldots, s_n)$ where each s_i is a semi-constructor term in $V_{\overline{M_C}^i(G)}$. Since there exists k ($\geq i$) such that $s_j\sigma \leftrightarrow^*_{\overline{M_C}^k(G)} s_j\theta$, we have $s\sigma \leftrightarrow^*_{M_F(\overline{M_C}^k(G))} s\theta$.

(3) Since G_{2_i} is convergent, there exists v with $\ell\sigma \to^*_{2_i} v \leftarrow^*_{2_i} u$. Here, $u \to^*_{2_i} v$ and $\ell\sigma \to^*_{2_i} v$ imply $u\ (\to_{2_i} \cap \overset{\varepsilon <}{\underset{R}{\to}})^*\ v$ and $\ell\sigma\ (\to_{2_i} \cap \overset{\varepsilon <}{\underset{R}{\to}})^*\ v$, respectively. Since R is non-E-overlapping, $\ell\sigma \to^*_{2_i} v$ has no reductions at $\mathrm{Pos}_F(\ell)$. By a similar argument to that of (1), we have $\ell|_p\sigma \leftrightarrow^*_{\overline{M_C}^i(G)} v|_p$ for each $p \in \mathrm{Pos}_X(\ell)$.
Let $x \in \mathrm{Var}(\ell)$. Since $\overline{M_C}^i(G)$ is convergent from Corollary 19 (3), we have $x\sigma = \ell\sigma|_p \to^*_{\overline{M_C}^i(G)} x\theta \leftarrow^*_{\overline{M_C}^i(G)} v|_p$ for each $p \in \mathrm{Pos}_{\{x\}}(\ell)$ by taking θ as $x\theta = x\sigma{\downarrow}_{\overline{M_C}^i(G)}$. Since ℓ is weakly shallow, by repeating (2) to each step in $v|_p \to^*_{\overline{M_C}^i(G)} x\theta$, there exists k with $v \leftrightarrow^*_{2_k} \ell\theta$. We have $u\ (\overset{\varepsilon <}{\underset{R}{\to}} \cap \leftrightarrow^*_{2_k})^*\ v\ (\overset{\varepsilon <}{\underset{R}{\to}} \cap \leftrightarrow^*_{2_k})^*\ \ell\theta$ by Proposition 21. □

6 Bottom-Up Construction of Convergent Reduction Graph

From now on, we assume that a TRS R is non-E-overlapping, non-collapsing, and weakly shallow. We show that R is confluent by giving a transformation of any R-reduction graph G_0 (possibly) containing a divergence into a convergent and subterm-closed R-reduction graph G_4 with $G_0 \sqsubseteq G_4$. The non-collapsing condition is used only in Lemma 27. Note that non-overlapping is not enough to ensure confluence as R_1 in Example 2. Now, we see an overview by an example.

Example 26. Consider R_2 in Example 13. Given $G_0 = \{f(g(c), c) \leftarrow f(c, c) \to f(c, g(c)) \overset{\varepsilon}{\to} g^3(c)\}$, we firstly take the subterm graph $\mathrm{sub}(G_0)$ and apply the transformation on it recursively to obtain a convergent and subterm-closed reduction graph G. In the example case, $\mathrm{sub}(G_0)$ happens to be equal to G in Example 13, and already satisfies the conditions. Secondly, we apply Lemma 22 on $M_F(G)$ and obtain G_1 in Example 2. As the next steps, we will merge the top edges T_1 in $G_0 \cup G$ into G_1, where $T_1 = \{f(c, g(c)) \overset{\varepsilon}{\to} g^3(c), c \overset{\varepsilon}{\to} g(c)\}$. Note that top edges in G is necessary for subterm-closedness. The union $G_1 \cup T_1$ is not, however, confluent in general. Thirdly, we remove unnecessary edges from T_1 by Lemma 27, and obtain T (in the example $T = T_1$). Finally, by

Lemma 28, we transform edges in T into S with modifying G_1 into $G_{1'}$ so that $G_4 = G_{1'}|_D \cup S \cup M_C(\overline{M_C}^{k'}(G))$ is confluent ($k' \geq k$). The resultant reduction graph G_4 is shown in Fig. 3, where the dashed edges are in S and some garbage vertices are not presented. (See Example 30 for details of the final step.)

$$
\begin{array}{ccccccc}
c & & f(c,c) & \rightarrow & f(g(c),c) & & f(g^2(c),c) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
g(c) & & f(c,g(c)) & \rightarrow & f(g(c),g(c)) & \rightarrow & f(g^2(c),g(c)) \\
\downarrow & & & & & & \downarrow \\
g^2(c) & & f(c,g^2(c)) & \rightarrow & f(g(c),g^2(c)) & \rightarrow & f(g^2(c),g^2(c)) \\
\downarrow & & & & & & \downarrow \\
g^3(c) & \rightarrow & g^4(c) & \rightarrow & g^5(c) & \leftarrow & f(g^2(c),g^3(c))
\end{array}
$$

Fig. 3. G_4 constructed by Lemma 29 from G_0 in Example 26

6.1 Removing Redundant Edges and Merging Components

For R-reduction graphs $G_1 = \langle V_1, \rightarrow_1 \rangle$ and $T_1 = \langle V_1, \rightarrow_{T_1} \rangle$, the *component graph* (denoted by T_1/G_1) of T_1 with G_1 is the graph $\langle \mathcal{V}, \rightarrow_{\mathcal{V}} \rangle$ having connected components of G_1 as vertices and $\rightarrow_{T_1}$ as edges such that

$$\mathcal{V} = \{[v]_{\leftrightarrow_1^*} \mid v \in V_1\}, \quad \rightarrow_{\mathcal{V}} = \{([u]_{\leftrightarrow_1^*}, [v]_{\leftrightarrow_1^*}) \mid (u,v) \in \rightarrow_{T_1}\}.$$

Lemma 27. *Let $G_1 = \langle V_1, \rightarrow_1 \rangle$ be an R-reduction graph obtained from Lemma 22, and let $T_1 = \langle V_1, \rightarrow_{T_1} \rangle$ be an R-reduction graph with $\rightarrow_{T_1} = \stackrel{\varepsilon}{\rightarrow}_{T_1}$. Then, there exists a subgraph $T = \langle V_1, \rightarrow_T \rangle$ of T_1 with $\rightarrow_T \subseteq \rightarrow_{T_1}$ that satisfies the following conditions.*

(1) $(\leftrightarrow_1 \cup \leftrightarrow_{T_1})^* = (\leftrightarrow_1 \cup \leftrightarrow_T)^*$.
(2) The component graph T/G_1 is acyclic in which each vertex has at most one outgoing-edge.

Proof. We transform the component graph T_1/G_1 by removing edges in cycles and duplicated edges so that preserving its connectivity. This results in an acyclic directed subgraph $T = \langle V_1, \rightarrow_T \rangle$ without multiple edges.

Suppose some vertex in T/G_1 has more than one outgoing-edges, say $\ell\sigma \rightarrow_T r\sigma$ and $\ell'\theta \rightarrow_T r'\theta$, where $\ell\sigma \leftrightarrow_1^* \ell'\theta$, $r\sigma, r\theta \in V_1$ and $\ell \rightarrow r, \ell' \rightarrow r' \in R$. Since R is non-E-overlapping, we have $\ell = \ell'$ and $r = r'$. By the condition ii) of Lemma 22, $\ell\sigma \leftrightarrow_{2_k}^* \ell\theta$ holds. Since R is non-collapsing, Lemma 25 (1) and (2) ensure $r\sigma \leftrightarrow_{2_j}^* r\theta$ for some j ($\geq k$). By the condition iii) of Lemma 22, $r\sigma \leftrightarrow_1^* r\theta$. These edges duplicate, contradicting to the assumption. □

In Lemma 27, if $\rightarrow_T$ is not empty, there exists a vertex of T/G_1 that has outgoing-edges, but no incoming-edges. We call such an outgoing-edge a *source edge*. Lemma 28 converts T to S in a source to sink order (by repeatedly choosing source edges) such that, for each edge in S, the initial vertex is $\rightarrow_1$-normal.

Lemma 28. *Let G_1, S, and T be R-reduction graphs, where G_1 and k satisfy the conditions i) to iv) of Lemma 22. Assume that the following conditions hold.*

v) $V_S = V_T = V_{G_1}$, $\to_S = \xrightarrow{\varepsilon}_S$, $\to_T = \xrightarrow{\varepsilon}_T$, *and* $\to_S \cap \to_T = \emptyset$.
vi) *The component graph* $(S \cup T)/G_1$ *is acyclic, where outgoing-edges are at most one for each vertex. Moreover, if* $[u]_{\leftrightarrow_1^*}$ *has an incoming-edge in* T/G_1 *then it has no outgoing-edges in* S/G_1.
vii) u *is* $\to_1$*-normal and* $u \not\leftrightarrow_1^* v$ *for each* $(u, v) \in \to_S$.

When $\to_T \neq \emptyset$, *there exists a conversion* $(S, T, G_1, k) \vdash (S', T', G_{1'}, k')$ *that preserves the conditions i) to iv) of Lemma 22, and conditions v) to vii), and satisfies the following conditions (1) to (3).*

(1) $G_{1'}$ *is a conservative extension of* G_1.
(2) $(\leftrightarrow_T \cup \leftrightarrow_S)^* \subseteq (\leftrightarrow_{T'} \cup \leftrightarrow_{S'} \cup \leftrightarrow_{1'})^*$.
(3) $|\to_T| > |\to_{T'}|$

Proof. We design $\vdash$ as sequential applications of $\vdash_\ell$, $\vdash_r$, and $\vdash_e$ in this order. We choose a source edge $(\ell\sigma, r\sigma)$ (of T/G_1) from T. We will construct a substitution θ such that $(\ell\sigma){\downarrow_1}\ (\underset{R}{\xrightarrow{\varepsilon\leq}} \cap \leftrightarrow_{2_{k'}}^*)^*\ \ell\theta$ and $(r\sigma){\downarrow_1}\ (\underset{R}{\xrightarrow{\varepsilon\leq}} \cap \leftrightarrow_{2_{k'}}^*)^* \cdot (\underset{R}{\xleftarrow{\varepsilon\leq}} \cap \leftrightarrow_{2_{k'}}^*)^*\ r\theta$ for enough large k'. The former sequence is added to G_1 by $\vdash_\ell$, the latter is added to G_1 by $\vdash_r$, and $\vdash_e$ removes $(\ell\sigma, r\sigma)$ from T and adds $(l\theta, r\theta)$ to S.

We have $\ell\sigma \to_1^* (\ell\sigma){\downarrow_1}$ by i), and $\ell\sigma \leftrightarrow_{2_k}^* (\ell\sigma){\downarrow_1}$ by ii). From Lemma 25 (3), there are $k^\ell \geq k$ and a substitution θ such that $x\sigma \to^*_{\overline{M_C}^k(G)} x\theta$ for each $x \in \mathrm{Var}(\ell)$, $(\ell\sigma){\downarrow_1} = u_0 \underset{R}{\xrightarrow{\varepsilon\leq}} u_1 \underset{R}{\xrightarrow{\varepsilon\leq}} \cdots \underset{R}{\xrightarrow{\varepsilon\leq}} u_n = \ell\theta$, and $u_{j-1} \leftrightarrow_{2_{k^\ell}}^* u_j$ for each $j (\leq n)$.

($\vdash_\ell$) We define $(S, T, G_1, k) \vdash_\ell (S, T, G_{1^\ell}, k^\ell)$ by $G_{1^\ell} = G_1 \multimap (u_0 \to \cdots \to u_n)$ to satisfy $(\ell\sigma){\downarrow_1} \leftrightarrow_{1^\ell}^* \ell\theta$ such that $\ell\theta$ is G_{1^ℓ}-normal. Since u_0 is $\to_1$-normal, the case (1) of Corollary 24 holds, so that $\vdash_\ell$ preserves i) to iv) for G_{1^ℓ} and k^ℓ. (1) and (2) are immediate. From (1), vi) is preserved. Since $[\ell\sigma]_{\leftrightarrow_1^*}$ does not have outgoing edges in S by vi), vii) is preserved.

($\vdash_r$) We define $(S, T, G_{1^\ell}, k^\ell) \vdash_r (S, T, G_{1'}, k')$. Let $G_{1^\ell} = \langle V_{1^\ell}, \to_{1^\ell} \rangle$. Since $x\sigma \leftrightarrow^*_{\overline{M_C}^{k^\ell}(G)} x\theta$ by Proposition 21 and $r\sigma \in V_{2_{k^\ell}}$, we obtain $r\sigma \leftrightarrow_{2_{k'}}^* r\theta$ for some $k' \geq k^\ell$ by Lemma 25 (2). We construct $G_{1'}$ to satisfy $(r\sigma){\downarrow_{1^\ell}} \leftrightarrow_{1'}^* r\theta$. Since the confluence of $G_{2_{k'}}$ follows from Corollary 19 (3) and Proposition 14 (1), we have the following sequences.

$$\begin{cases} (r\sigma){\downarrow_{1^\ell}} = u_0 \to_{2_{k'}} u_1 \to_{2_{k'}} \cdots \to_{2_{k'}} u_n = v, \\ \quad r\theta \quad\;\; = v_0 \to_{2_{k'}} v_1 \to_{2_{k'}} \cdots \to_{2_{k'}} v_m = v, \end{cases}$$

where we choose the least n satisfying $u_n = v_m$. There are two cases according to the second sequence.

(a) If $v_i \in V_{1^\ell}$ for some i, we choose i as the least. If $i = 0$, then $G_{1'} = G_{1^\ell}$. Otherwise, let $G_{1'} := G_{1^\ell} \multimap (v_0 \to v_1 \to \cdots \to v_i)$. Since G_{1^ℓ} satisfies the case (2) of Corollary 24, $\vdash_r$ preserves i) to iv). Since $u_0 \leftrightarrow_{2_{k'}}^* v_i$ and $u_0, v_i \in V_{1^\ell}$, $u_0 \leftrightarrow_{1^\ell}^* v_i$ by iii). Thus, $(r\sigma){\downarrow_{1^\ell}} \leftrightarrow_{1'}^* r\theta$.

(b) Otherwise (i.e., $v_i \notin V_{1^\ell}$ for each i), let
$$\begin{cases} G_{1''} := G_{1^\ell} \multimap (u_0 \to u_1 \to \cdots \to u_n) \\ G_{1'} := G_{1''} \multimap (v_0 \to v_1 \to \cdots \to v_m). \end{cases}$$

Since u_0 is G_{1^ℓ}-normal and $u_j \in V_{1^\ell}$ implies $u_0 \leftrightarrow^*_{1^\ell} u_j$ (by iii) of G_{1^ℓ}, $G_{1''}$ and k' satisfy i) to iv) by Corollary 24. Let $G_{1''} = \langle V_{1''}, \rightarrow_{1''} \rangle$. Since $v_i \notin V_{1''}$ for each i ($< m$) and $v_m = u_n = v \in V_{1''}$, $G_{1'}$ and k' also satisfy i) to iv) by Corollary 24. By construction, $(r\sigma){\downarrow_{1^\ell}} \leftrightarrow^*_{1'} r\theta$ holds.

Since S and T do not change, $\vdash_r$ keeps v), (1), and (2). Lastly, vi) and vii) follows from (1).

($\vdash_e$) We define $(S, T, G_{1'}, k') \vdash_e (S', T', G_{1'}, k')$, where $V_{S'} = V_{G_{1'}}$, $V_{T'} = V_{G_{1'}}$, $\rightarrow_{S'} = \rightarrow_S \cup \{(\ell\theta, r\theta)\}$, and $\rightarrow_{T'} = \rightarrow_T \setminus \{(\ell\sigma, r\sigma)\}$. Since $(\ell\sigma, r\sigma)$ is a source edge of T/G_1, $\vdash_e$ preserves vi). Conditions i) to v), (1) and (3) are trivial. Since $\ell\sigma \leftrightarrow^*_{G_{1'}} (\ell\sigma){\downarrow_1} \leftrightarrow^*_{G_{1'}} \ell\theta \rightarrow_{S'} r\theta \leftrightarrow^*_{G_{1'}} (r\sigma){\downarrow_{1^\ell}} \leftrightarrow^*_{G_{1'}} r\sigma$ implies $(\ell\sigma, r\sigma) \in \leftrightarrow^*_{S' \cup G_{1'}}$, we have (2). vii) holds from vi). ☐

6.2 Construction of a Convergent and Subterm-Closed Graph

Lemma 29. *Let $G_0 = \langle V_0, \rightarrow_0 \rangle$ be an R-reduction graph. Then, there exists a convergent and subterm-closed R-reduction graph G_4 with $G_0 \sqsubseteq G_4$.*

Proof. By induction on the sum of the size of terms in V_0, i.e., $\Sigma_{v \in V_0} |v|$. If G_0 has no vertex, we set $G_4 = G_0$, which is the base case. Otherwise, by induction hypothesis, we obtain a convergent and subterm-closed R-reduction graph G with $\mathrm{sub}(G_0) \sqsubseteq G$. We refer to the conditions i) to vii) in Lemma 28.

Let $G_1 = \langle V_1, \rightarrow_1 \rangle$ and k be as in Lemma 22. Let T be obtained from G_1 and $T_1 = \langle V_1, \rightarrow_{G^\varepsilon} \cup \rightarrow_{G_0^\varepsilon} \rangle$ by applying Lemma 27.

Let $S = \langle V_1, \emptyset \rangle$. For G_1 and k, i) to iv) hold by Lemma 22. vi) holds by Lemma 27 (2) and $\rightarrow_S = \emptyset$, and vii) trivially holds. Starting from (S, T, G_1, k), we repeatedly apply $\vdash$ (in Lemma 28), which moves edges in T to S until $\rightarrow_T = \emptyset$. Finally, we obtain $(S', \langle V_{1'}, \emptyset \rangle, G_{1'}, k')$ that satisfies i) to vii) and (1) to (3) in Lemma 28, where $G_{1'} = \langle V_{1'}, \rightarrow_{1'} \rangle$ and $V_{S'} = V_{1'}$. From Lemmas 27 and 28 (1) and (2), $(\leftrightarrow_1 \cup \leftrightarrow_{G^\varepsilon} \cup \leftrightarrow_{G_0^\varepsilon})^* = (\leftrightarrow_1 \cup \leftrightarrow_T)^* \subseteq (\leftrightarrow_{1'} \cup \leftrightarrow_{S'})^*$. Note that $G_{1'}$ is convergent by i).

Let $G_3 = \langle V_3, \rightarrow_3 \rangle$ be $S' \cup G_{1'}$. This is obtained by repeatedly extending $G_{1'}$ by $G_{1'} \multimap (u \rightarrow v)$ for each $(u, v) \in \rightarrow_{S'}$, since in each step vii) is preserved; u is $\rightarrow_{1'}$-normal and $u \not\leftrightarrow^*_{1'} v$. Thus, the convergence of G_3 follows from Proposition 7.

We show $G_0 \sqsubseteq G_3$. Since $G_0^\varepsilon \subseteq T_1 \sqsubseteq G_1 \cup T \sqsubseteq G_{1'} \cup S'$ (by Lemmas 27 and 28) and $M_F(\mathrm{sub}(G_0)) \sqsubseteq M_F(G) \sqsubseteq G_1 \sqsubseteq G_{1'}$ (by $\mathrm{sub}(G_0) \sqsubseteq G$ and iv)), $G_0 \subseteq G_0^\varepsilon \cup M_F(\mathrm{sub}(G_0)) \sqsubseteq S' \cup G_{1'} = G_3$.

Let $G_4 = \langle V_4, \rightarrow_4 \rangle$ be given by $G_4 := G_3|_D \gtrdot M_C(\overline{M_C}^{k'}(G))$. We show $G_0 \sqsubseteq G_4$ by showing $G_3 \sqsubseteq G_4$. Since $G_{1'} \sqsubseteq G_{2_{k'}}$ by ii) where $G_{2_{k'}}$ contains no top edges, we have $V_{1'}|_C \subseteq V_{2_{k'}}|_C$ and $\rightarrow_{1'}|_C \subseteq (\leftrightarrow_{2_{k'}}|_C)^*$. Since $\rightarrow_{2_{k'}}|_C = \rightarrow_{M_C(\overline{M_C}^{k'}(G))}$, we have $G_{1'}|_C \sqsubseteq \langle V_{1'}, \emptyset \rangle \cup M_C(\overline{M_C}^{k'}(G))$. Thus, $G_{1'} = G_{1'}|_D \cup G_{1'}|_C \sqsubseteq G_{1'}|_D \cup M_C(\overline{M_C}^{k'}(G))$. By $S' = S'|_D$, we have $G_3 = S' \cup G_{1'} \sqsubseteq S'|_D \cup G_{1'}|_D \cup M_C(\overline{M_C}^{k'}(G)) = G_4$.

Now, our goal is to show that G_4 is convergent and subterm-closed. The convergence of $G_4 = G_3|_D \gtrdot M_C(\overline{M_C}^{k'}(G))$ is reduced to that of $G_3 = G_3|_D \gtrdot$

$\langle V_3|_C, \to_3|_C\rangle$ by Proposition 4 and Lemma 5. Their requirements are satisfied from $\langle V_3|_C, \to_3|_C\rangle = \langle V_{1'}|_C, \to_{1'}|_C\rangle \sqsubseteq M_C(\overline{M_C}^{k'}(G))$ by ii) and the convergence of $M_C(\overline{M_C}^{k'}(G))$ by Corollary 19 (3) and Proposition 14 (1).

We will prove that G_4 is subterm-closed by showing $\mathrm{sub}(G_4^{\neq\varepsilon}) \sqsubseteq \overline{M_C}^{k'}(G)$ and $\overline{M_C}^{k'}(G) \sqsubseteq G_4$. Note that $\mathrm{sub}(G_4^{\neq\varepsilon}) = \mathrm{sub}((S'|_D)^{\neq\varepsilon} \cup (G_{1'}|_D)^{\neq\varepsilon} \cup (M_C(\overline{M_C}^{k'}(G)))^{\neq\varepsilon}) \subseteq \mathrm{sub}(S'^{\neq\varepsilon}) \cup \mathrm{sub}(G_{1'}|_D) \cup \overline{M_C}^{k'}(G)$. We have $\mathrm{sub}(S'^{\neq\varepsilon}) = \langle \mathrm{sub}(V_{1'}), \emptyset\rangle$. Since $G_{2_{k'}}$ has no top edges and $G_{1'} \sqsubseteq G_{2_{k'}}$ by ii), $\mathrm{sub}(G_{1'}) \sqsubseteq \mathrm{sub}(G_{2_{k'}}) = \mathrm{sub}(M_F(\overline{M_C}^{k'}(G))) \subseteq \overline{M_C}^{k'}(G)$. Thus, $\mathrm{sub}(G_4^{\neq\varepsilon}) \sqsubseteq \overline{M_C}^{k'}(G)$.

It remains to show $\overline{M_C}^{k'}(G) \sqsubseteq G_4$, which is reduced to $G|_D \sqsubseteq G_4$ from $\overline{M_C}^{k'}(G) = G|_D \cup M_C(\overline{M_C}^{k'-1}(G))$, $M_C(\overline{M_C}^{k'}(G)) \subseteq G_4$, and Proposition 21. Since $G|_D \subseteq G \sqsubseteq G^\varepsilon \cup M_F(G)$ by Lemma 17 (2), it is sufficient to show that $G^\varepsilon \sqsubseteq G_4$ and $M_F(G) \sqsubseteq G_4$.

Obviously, $M_F(G) \sqsubseteq G_{1'} \subseteq G_3 \sqsubseteq G_4$ holds, since $M_F(G) \sqsubseteq G_{1'}$ by iv). We show $G^\varepsilon \sqsubseteq G_4$. Since $V_G \subseteq V_{M_F(G)}$ by Proposition 14 (2), we have $V_{G^\varepsilon} = V_G \subseteq V_{M_F(G)} \subseteq V_{1'} \subseteq V_3 \subseteq V_4$. By Lemmas 27 (1) and 28 (2), $\to_{G^\varepsilon} \subseteq (\leftrightarrow_{G_{1'}} \cup \leftrightarrow_{S'})^*$ holds, and by ii) we have $\to_{G_{1'}|_C} \subseteq \leftrightarrow^*_{M_C(\overline{M_C}^{k'}(G))}$. Hence, $\to_{G^\varepsilon} \subseteq (\leftrightarrow_{G_{1'}|_D} \cup \leftrightarrow_{S'} \cup \leftrightarrow_{M_C(\overline{M_C}^{k'}(G))})^* = \leftrightarrow^*_{G_4}$. Therefore G_4 is subterm-closed. □

Example 30. Let us consider applying Lemma 29 on G_1 and T in Example 26, where $k = 1$. The edge $c \to g(c)$ in T is simply moved to S. For the edge $f(c, g(c)) \to g^3(c)$ in T, $\vdash_\ell$ adds $f(g^2(c), g^2(c)) \to f(g^2(c), g^3(c))$ to G_1. $\vdash_r$ adds $g^3(c) \to g^4(c) \to g^5(c)$ to G_1 and increases k to 3. $\vdash_e$ adds $f(g^2(c), g^3(c)) \to g^5(c)$ to S. Since $M_C(\overline{M_C}^3(G))$ is $\{g(c) \to g^2(c) \to \cdots \to g^4(c) \to g^5(c),\quad g^6(c)\}$, $G_4 = (S \cup G_1|_D) \gtrdot M_C(\overline{M_C}^3(G))$ is as in Fig. 3.

Theorem 31. *Non-E-overlapping, weakly shallow, and non-collapsing TRSs are confluent.*

Proof. Let $u \leftarrow^*_R s \to^*_R t$. We obtain G_4 by applying Lemma 29 to an R-reduction graph G_0 consisting of the sequence. By $G_0 \sqsubseteq G_4$ and the convergence of G_4, $u{\downarrow}_{G_4} = t{\downarrow}_{G_4}$. Thus we have $u \to^*_R s' \leftarrow^*_R t$ for some s'. □

Corollary 32. *Strongly non-overlapping, weakly shallow, and non-collapsing TRSs are confluent.*

7 Conclusion

This paper extends the reduction graph technique [SO10] and has shown that *non-E-overlapping, weakly shallow, and non-collapsing TRSs are confluent.*

We think that the *non-collapsing* condition can be dropped by refining the reduction graph techniques. A further step will be to relax the *weakly shallow* to the *almost weakly shallow* condition, which allows at most one occurrence of a defined function symbol in each path from the root to a variable.

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