On the Uniquely Converging Property of Nonlinear Term Rewriting Systems

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Abstract

A *uniquely converging* (**UC**) property for a possibly nonlinear term rewriting system (TRS) is investigated. **UC**, which is an intermediate property between conventional *Church-Rosser* (**CR**) and *uniquely normalizing* (**UN**), is newly proposed in connection with the consistency of *continuous semantics*. *Continuous semantics* is defined by constructing *free-continuous algebra* which is required in algebraic specification on a lazy space. In fact, *free-continuous algebra* can specify a lazy space, whereas neither *initial algebra* nor *final algebra* can.

This paper also clarifies a sufficient condition for **UC**. The statement is, *an ω-nonoverlapping TRS is* **UC** (irrespective of linearlity). This makes the contrast with the well-known facts that a nonoverlapping TRS is possibly non-**UN** when nonlinear, although **CR** when left linear. The difference between *ω-nonoverlapping* and usual nonoverlapping is that *unification with infinite terms* is applied instead of usual unification with occur-check.

1 Introduction

A Term Rewriting System (TRS), intuitively which is a set of directed equations (reduction rules), have been applied as a model for representing computational processes of equational logic and algebraic specification [9]. As theoretical foundation, its declarative semantics have been investigated in several literatures [1, 6, 11, 14]. The method mainly depends on algebraic semantics, that is, to construct an algebra corresponding to a given TRS. In other words, which and which should be specified equal.

The algebraic semantics is quite clear under the assumption of termination. For instance, let us count an initial algebra and a final algebra, which give the pair of the most detailed and the most abstract semantics. Intuitively speaking, equality in the initial algebra is defined to be $MustEqual(x,y) \stackrel{\text{def}}{=} \{\ x = y$ is deduced from E in finite steps $\}$ (else x and y are naturally assumed to be unequal), where reduction rules in R are interpreted as equational deduction rules. In contrast, inequality in the final algebra is defined to be $CannotEqual(x,y) \stackrel{\text{def}}{=} \{\ x \neq y$ is deduced from $E \cup \{\textbf{true} \neq \textbf{false}\}$ in finite steps $\}$ (else x and y are naturally assumed to be equal). These two equalities will coincide on terminating computations (if $CannotEqual$ is well-defined) [9]. Differences may be found among erroneous computations, such as nontermination.

In turn, once we take account into lazy-evaluation, a nonterminating computation becomes the center of interest. However, neither initial algebra specification nor final algebra specification can specify lazy space [3]. For instance, let us examine **example 1** [16]. The example shows that nontermination on a lazy space is classified into two cases : diverging as an error (e.g.$h(x)$), and generating an infinite data structure (e.g.$intseq(x)$,$intseq'(x)$ which generate an infinite increasing sequence starting from given x). Initial algebra distinguishes each cases and final algebra identifies all cases, but should be $intseq(x) = intseq'(x) \neq h(x)$.

Example 1

$$R_1 \stackrel{\text{def}}{=} \left\{ \begin{array}{lcl} intseq(x) & \rightarrow & cons(x, intseq(s(x))) \\ intseq'(x) & \rightarrow & cons(x, intseq'(s(x))) \\ h(x) & \rightarrow & h(h(x)) \end{array} \right\}$$

Therefore, the semantics of an infinite object, which is defined as the limit point of a sequence of finite approximations, requires some kind of the least fix point operation (as in *denotational semantics* of a functional language). Then, as a natural way, definedness-ordering is induced from a TRS. That is, the more a reduction proceeds, the more a term becomes *informative* (still omitting diverged computations). This method, called *continuous semantics*, is based on the construction of *free-continuous algebra* [1, 6, 11, 14]. This semantics have been investigated dependent on *Church-Rosser* (**CR**) property of a nonover-

Figure 1: A nonoverlapping, but not $\mathbf{UN}^{\rightarrow}$ example R_2.

lapping TRS. From this limitation, the objective TRSs are restricted on a left linear TRS, though a nonlinear TRS is the first important step to describe equality among infinite objects. For instance, **Example 2** shows that a nonoverlapping and nonlinear TRS is not *uniquely normalizing* ($\mathbf{UN}^{\rightarrow}$) in general (See Figure 1), though a nonoverlapping and left linear TRS is known to be *Church-Rosser* [8].

Example 2

$$R_2 \stackrel{\text{def}}{=} \left\{ \begin{array}{lll} d(x,x) & \rightarrow & 0 \\ d(x,f(x)) & \rightarrow & 1 \\ 2 & \rightarrow & f(2) \end{array} \right\}$$

In this paper, a *uniquely converging* (**UC**) property for a possibly nonlinear TRS is investigated. **UC**, which is an intermediate property between conventional **CR** and **UN**, is newly proposed in connection with the consistency of *continuous semantics*. Further, a sufficient condition for **UC** is clarified. The statement is, *an ω-nonoverlapping TRS is* **UC**. The difference between *ω-nonoverlapping* and usual nonoverlapping conditions is that *unification with infinite terms* is applied instead of usual unification with occur-check.

In section 2, **UC** property is formally defined in terms of an abstract reduction system. The relation among these **CR**-related properties is also investigated. In section 3, the relation between *continuous semantics* and **UC** property is discussed. A sufficient but undecidable condition for **UC** property is also proposed. The statement is, *an E-nonoverlapping TRS is* **UC**, where *E-nonoverapping* is intuitively the nonoverlapping condition under *modulo* an associated equational logic E. In section 4, a decidable condition for *E-nonoverlapping* property is proposed. The statement is, *an ω-nonoverlapping TRS is E-nonoverlapping.* Thus, an *ω-nonoverlapping* TRS is proved to be **UC**. This result is also compared with classical results found in [4, 10].

2 Reduction systems

2.1 Abstract reduction systems

A *reduction system* is a structure $R = \langle A, \rightarrow \rangle$ consisting of an object set A and any binary relation $\rightarrow$ on A (i.e. $\rightarrow \subseteq A \times A$), called a *reduction relation.* A *reduction* (starting with x_0) in R is a finite or an infinite sequence $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots$. The transitive closure of $\rightarrow$ is noted as $\xrightarrow{*}$. A sequence $x \equiv x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_n \equiv y$ is said to be a *reduction-path* $\Re_{x \xrightarrow{*} y}$.

An *equational system* associated to a reduction system $R = \langle A, \rightarrow \rangle$ is a structure $E = \langle A, \doteq_R \rangle$ (or simply $E = \langle A, \doteq \rangle$) consisting of A and the symmetric binary relation $\doteq_R$ (or simply $\doteq$) which is defined to be $x \doteq_R y \Longleftrightarrow (x \rightarrow y \ \vee \ y \rightarrow x)$. An *equality* $=_R$ (or simply $=$) in E is the transitive reflexive closure of the binary relation $\doteq_R$. A sequence $x \equiv x_0 \doteq x_1 \doteq x_2 \doteq \cdots \doteq x_n \equiv y$ is said to be an *equality-path* $\Im_{x=y}$.

A *combination* of equality-pathes $\Im_{x=y}$ and $\Im_{y=z}$ is denoted as $\Im_{x=y} \cdot \Im_{y=z}$. A *step* of an equality-path $\Im_{x=y}$ is denoted as $\#\Im_{x=y}$. For a reduction path $\Re_{x \xrightarrow{*} y}$, the combination and the step are similarly defined by treating $\Re_{x \xrightarrow{*} y}$ as an equality-path.

Definition A set of *normal forms* of R is defined as $NF(R) \stackrel{\text{def}}{=} \{x \in A \mid \neg \exists y \text{ s.t. } x \rightarrow y\}$

Definition Let $R = \langle A, \rightarrow \rangle$ be a reduction system. Assume D be a base domain such that $A \subseteq D$. Then, $\psi : D \rightarrow D$ is said to be a *normal retraction* iff

- $\psi \circ \psi(x) = \psi(x)$ for $\forall x \in D$.
- $\psi(x) = x$ for $\forall x \in NF(R)$.
- $(\psi(D), \sqsubseteq)$ is an *algebraic cpo*[1]

R is said to be *monotonic* (*with respect to* ψ) iff

- ($x \rightarrow y \implies \psi(x) \sqsubseteq \psi(y)$) for $\forall x, y \in A$
- $x \in NF(R) \implies$ ($\psi(x) \not\sqsubseteq \psi(y)$ for $\forall y \in D$)

If ψ is a *normal retraction* and R is *monotonic with respect to* ψ, then ψ is said to be a *regular retraction*.

[1] Short for, algebraic complete partial order (See ch.1 in [2])

2.2 Hierarchy of Church-Rosser related properties

Church-Rosser related properties guarantee the validity of reduction-based computations in various levels. Intuitively speaking, *Church-Rosser* means that equality of terms may be examined without *back-track*. *Uniquely converging* means that the result of the computation is uniquely specified even for infinite computations. *Uniquely normalizing* means that the result of the computation is uniquely determined if terminates.

Definition [8] $R = \langle A, \rightarrow \rangle$ is said to be *Church-Rosser* (**CR**) iff $\forall x, y \in A$ s.t. $x = y \Longrightarrow x \downarrow y$ (i.e. $\exists z \in A$ s.t. $x \xrightarrow{*} z$ and $y \xrightarrow{*} z$).

Notation $\Theta_\psi(x) \stackrel{\text{def}}{=} \{\psi(y) \mid x = y\}$ ($\Theta(x)$, for short.)

Definition Let $R = \langle A, \rightarrow \rangle$ be a reduction system, and $\psi : D \rightarrow D$ be a regular retraction where $A \subseteq D$. R is said to be *uniquely converging (for ψ) with respect to equality* ($\mathbf{UC}_\psi$) iff $\Theta_\psi(x)$ is a directed set[2] for $\forall x \in A$.

Definition [13] $R = \langle A, \rightarrow \rangle$ is said to be *uniquely normalizing with respect to equality* (**UN**) iff $\forall x, y \in NF(R)$ s.t. $x =_R y \Longrightarrow x \equiv y$. ($x \equiv y$ iff x and y are syntactically same.)

$R = \langle A, \rightarrow \rangle$ is said to be *uniquely normalizing with respect to reduction* ($\mathbf{UN}^{\rightarrow}$) iff $\forall x \in A\ \forall y, z \in NF(R)$ s.t. $x \xrightarrow{*} y \wedge x \xrightarrow{*} z \Longrightarrow y \equiv z$.

Remark Let ψ be a regular retraction. Then, the logical relation among them is,

$$\mathbf{CR} \Longrightarrow \mathbf{UC}_\psi \Longrightarrow \mathbf{UN} \Longrightarrow \mathbf{UN}^{\rightarrow}$$

However, the converses are not satisfied in general (See Figure 2). If R is *weakly normalizing* (i.e. $\forall x \in A\ \exists y \in NF(R)$ s.t. $x \xrightarrow{*} y$), all these properties are equivalent.

Lemma Let $R = \langle A, \rightarrow \rangle$ be a $\mathbf{UC}_\psi$ reduction system, and $\psi : D \rightarrow D$ be a regular retraction where $A \subseteq D$. If $x \in A$ satisfies $\Theta(x) \cap \psi(NF(R)) \neq \phi$, then $\{lub(\Theta(x))\} = \Theta(x) \cap NF(R)$. (Thus, **UN**.)

[2]A set S is *directed* iff for every finite subset $U \subseteq S$, S contains an upper bound for U [2].

Figure 2: Relation among **CR**-related properties.

3 Continuous semantics of a TRS

3.1 Term rewriting systems

Term rewriting systems are reduction systems which has a term set $T(F, V)$ as an object set A. A term set $T(F, V)$ is a set of terms where F is a set of function symbols and V is a set of variable symbols. 0-ary function symbols are also called *constants*. $T(F, V)$ may be abbreviated as simply T. The substitution θ is a map from V to $T(F, V)$ such that θ is an identity map except on a finite number of variables. The syntactical equivalence between terms M and N is denoted as $M \equiv N$.

The context C is a term in $T(F \cup \{\Box\}, V)$ where $\Box$ is a special constant named a *hole*. The notation $C[N_1, \cdots, N_n]$ is a syntax convention for the result of placing $N_1, \cdots, N_n$ in the holes of $C[, \cdots,]$ from left to right. Then, N is said to be a *subterm* of M iff $M \equiv C[N]$ for some context C having a precisely one hole. The context $C[, \cdots,]$ is said to be *trivial* iff $C[\ ] \equiv \Box$.

Definition A finite set $R = \{(\alpha_i, \beta_i)\}$ of ordered pairs of two terms is said to be a *Term Rewriting System* (*TRS*) iff each α_i is not a variable and all variables in β_i appear in α_i. A *reduction* is defined on a term M as $M \rightarrow N$ iff there exists a context $C[\ ]$ and a substitution θ s.t. $M \equiv C[\theta(\alpha_i)]$ and $N \equiv C[\theta(\beta_i)]$. A subterm $M' \equiv \theta(\alpha_i)$ in M is said to be a *redex* (short for a *reducible expression*).

Definition A pair of reduction rules $\alpha_i \rightarrow \beta_i$ and $\alpha_j \rightarrow \beta_j$ is said to be *overlapping* iff there exists a context $C[\ ]$, a nonvariable term M, and a substitution θ s.t. $\alpha_i \equiv C[M]$ and $\theta(\alpha_j) \equiv \theta(M)$ (i.e. α_j and M are unifiable).

A TRS R is said to be *nonoverlapping* iff no pair of two rules in R are overlapping except trivial cases (i.e.

$i = j \wedge C[\,] \equiv \Box$).

Definition A reduction rule $\alpha_i \rightarrow \beta_i$ is said to be *left linear* iff any variable in α_i appears precisely once in α_i. A TRS R is said to be *left linear* iff all reduction rules in R are *left linear*. A TRS R is said to be *nonlinear* iff R is not left linear.

Remark A left linear nonoverlapping TRS is known to be confluent [8].

3.2 Continuous semantics and UC-property

Intuitively speaking, *continuous semantics* of a TRS R is an interpretation $Val_R : T(F,X) \rightarrow D$ such that $Val_R(x) = lub(\{\omega_R(y) \mid x =_E y\})$ where ω_R is an embedding into an algebraic cpo $(D, \sqsubseteq)$. For this purpose, there must be clarified following two points.

- How is ω_R defined ? (i.e. How is an algebraic cpo $(D, \sqsubseteq)$ constructed ?)
- Does *lub* exist ? (i.e. Is an interpretation Val_R well-defined ?)

It may be natural to introduce the ordering $\sqsubseteq$ as $x \sqsubseteq y \iff x \rightarrow_R y$. That is, the more a reduction proceeds, the more a term becomes *informative*. However, this idea is little bit too naive; some reductions may be redundant or fall into an idle loop, and some reductions may not terminate but generate infinite terms. Thus, the former requires a special constant $\perp$ which means *undefined*, and the latter requires an infinite term which means a *limit* of an approximation-sequence (consists of finite/infinite trees). For these purpose, a set of infinite trees $T^\infty(F \cup \{\perp\}, X)$, which is the completion of $T(F,X)$ for a *lub* (*least upper bound*) operation, is applied as an algebraic cpo D.

Definition *Definedness ordering* $\sqsubseteq$ is defined to be

$T \sqsubseteq T' \iff T$ is obtained from T' by replacing subtrees of T' with $\perp$.

for $\forall T, T' \in T^\infty(F \cup \{\perp\}, X)$.

Let $U \subseteq T^\infty(F \cup \{\perp\}, X)$. A pair of trees T_1, T_2 are said to be *cooperative in* U iff there exists $T \in U$ s.t. $T_1, T_2 \sqsubseteq T$. A pair of trees T_1, T_2 are said to be *individual in* U iff T_1 and T_2 are not *cooperative* in U.

Note that $T^\infty(F \cup \{\perp\}, X)$ is an algebraic cpo under the definedness ordering $\sqsubseteq$ [14]. Before defining a retraction $\omega_R : T^\infty(F \cup \{\perp\}, X) \rightarrow T^\infty(F \cup \{\perp\}, X)$ (which is also an embedding $\omega_R : T(F,X) \rightarrow T^\infty(F \cup \{\perp\}, X)$), several tree-related notations are introduced.

Definition An *occurrence* $occur(M,N)$ of a subterm N in a term M is defined inductively as

$$occur(M,N) \stackrel{\text{def}}{=} \begin{cases} \epsilon & \text{if } N \equiv M \\ i \cdot u & \text{if } u = occur(N_i, N) \text{ and } \\ & \quad M = f(N_1, \cdots, N_n) \end{cases}$$

The subterm N of M at occurrence u is denoted as M/u. (That is, $u = occur(M,N)$.) $Node(M)$ is a set of all occurrences in M (including a root occurrence ϵ). $Node^{\times}(M)$ is a set of all non-variable occurrences in M (i.e. $\{u \in Node(M) \mid M/u$ is not a variable$\}$).

Definition A *replacement* is noted as $T[u \leftarrow T']$ which is the replacement T/u with T' where u is an occurrence u in T. A *substitution* is noted as $T_{x \leftarrow T'} \stackrel{\text{def}}{=} T[u \leftarrow T' \mid \forall u \in occur(T,x)]$ for a variable x.

Then, a set of candidates of redexes $Cand_R$ is defined inductively as

- If $T \in Red_R$, then $T \in Cand_R$.
- If $T, T' \in Cand_R$, then $T[u \leftarrow T'] \in Cand_R$ for some occurrence u in T.

where Red_R is a set of all *redexes* of R.

Let $Cand_R^-$ be a closure of $Cand_R$ under *Scott topology* [2] on the algebraic cpo $(T^\infty(F \cup \{\perp\}, X), \sqsubseteq)$. A set of occurrences of $Cand_R^-$ which appears in a term M is noted as

$$Candocc_R(M) \stackrel{\text{def}}{=} \{u \in Node(M) \mid M/u \in Cand_R^-\}.$$

Definition The order on occurrences u, v is defined as $u \preceq v \iff \exists w$ s.t. $v = u \cdot w$. If $u \preceq v \wedge u \neq v$ then it is noted as $u \prec v$. The occurrences u, v is said to be *disjoint* and noted $u|v$ iff $u \not\preceq v$ and $v \not\preceq u$.

Let U be any set of occurrences. A set of minimum occurrences in U is noted as

$$Min(U) \stackrel{\text{def}}{=} \{u \in U \mid v \not\prec u \text{ for } \forall v \in U\}.$$

Definition The retraction $\omega_R : T^\infty(F \cup \{\perp\}, X) \rightarrow T^\infty(F \cup \{\perp\}, X)$ and the interpretation $Val_R^{\rightarrow}, Val_R : T(F,X) \rightarrow T^\infty(F \cup \{\perp\}, X)$ are defined to be

$$\begin{cases} \omega_R(M) \stackrel{\text{def}}{=} M[\, u \leftarrow \perp \mid \forall u \in Min(Candocc_R(M)) \,] \\ Val_R^{\rightarrow}(M) \stackrel{\text{def}}{=} lub(\{\omega_R(N) \mid M \stackrel{*}{\rightarrow} N\}) \\ Val_R(M) \stackrel{\text{def}}{=} lub(\{\omega_R(N) \mid M =_R N\}) \end{cases}$$

Note that the retraction ω_R is regular [14].

Definition A TRS R is said to be **UC** iff $\mathbf{UC}_{\omega_R}$.

The value of a term M in *continuous semantics* of a TRS is given as $Val_R(M)$. Thus, the well-definedness of Val_R is equivalent to **UC** property. Adding to it, **UC** property implies the continuity of Val_R. For detailed discussions on the continuous semantics, refer [14].

Remark Note that $Val_{\overrightarrow{R}}(M)$ is not well-defined even if R is **UC**. Further, they are generally unequal (i.e. $Val_{\overrightarrow{R}}(M) \sqsubseteq Val_R(M)$), though they are well-defined and coincide if R is **CR**.

Example 3

$$R_3 \stackrel{\text{def}}{=} \left\{ \begin{array}{lcl} 1 & \rightarrow & f(1) \\ h(x) & \rightarrow & h(h(x)) \\ d_1(x,x) & \rightarrow & cons(x,x) \\ d_1(x,f(x)) & \rightarrow & d_2(x,x) \\ d_2(x,x) & \rightarrow & cons(x,h(x)) \\ d_2(x,f(x)) & \rightarrow & cons(h(x),x) \end{array} \right\}$$

In fact, R_3 is **UC**, but $Val_{\overrightarrow{R}}(d_2(1,1))$ is not well-defined. If the last rule $d_2(x,f(x)) \rightarrow cons(h(x),x)$ was removed, $Val_{\overrightarrow{R}}$ becomes well-defined, but still $cons(1,\bot) \equiv Val_{\overrightarrow{R}}(d_2(1,1)) \sqsubset Val_R(d_2(1,1)) \equiv cons(1,1)$.

In the following sections, sufficient conditions for **UC**-property will be investigated.

3.3 UC-property of an E-nonoverlapping TRS

In this section, the sufficient condition for **UC** property in terms of nonoverlapping property is introduced. Intuitively speaking, a TRS R is said to be *E-nonoverlapping* iff R is nonoverlapping *modulo* an associated equational logic E.

Definition An occurrence u is said to be *invariant* in the equality-path $\Im_{M=N}$ iff $v \not\preceq u$ for any occurrence v at which some reduction in $\Im_{M=N}$ occurs. A set of all invariant occurrences in the equality-path $\Im_{M=N}$ is noted as $O_{inv}(\Im_{M=N})$.

Definition Let R be a TRS. A pair of reduction rules $\alpha_i \rightarrow \beta_i$ and $\alpha_j \rightarrow \beta_j$ is said to be *E-overlapping* iff there exist a context $C[\ ]$, a nonvariable term M, and a substitution θ s.t. $\alpha_i \equiv C[M]$, and $(\theta(\alpha_j) =_R \theta(M)) \wedge (\epsilon \in O_{inv}(\Im_{\theta(\alpha_j)=\theta(M)}))$.

A TRS R is said to be *E-nonoverlapping* iff no pair of two rules in R are E-overlapping except trivial cases (i.e. $i = j \wedge C[\ ] \equiv \Box$).

Theorem 1 An *E-nonoverlapping* TRS R is **UC**.

The proof consists of three steps. The first step, a key lemma **normalization lemma** is introduced. The next, an E-nonoverlapping TRS is proved to be **UN**. Finally, an E-nonoverlapping TRS is proved to be **UC**. (See **Appendix A**.)

4 A sufficient condition for E-nonoverlapping property

4.1 Unification with infinite terms

Unifications are classified into following three classes. They are,

- Unification without occur check.
- Unification with occur check.
- Unification with infinite terms (called *infinite unification*).

Unification without occur check does not care on name conflicts. Thus, even for finite terms, this is not correct for nonlinear terms. For instance, $f(x,x)$ and $f(g(y),h(y))$ are unified as $\{x = g(y),\ x = h(y)\}$. In other words, consistency of binding environments is not preserved.

In contrast, unification with occur check treats name conflicts as unification *failed*. This is correct on finite terms, but not correct on infinite terms. For instance, unification between $f(x,x)$ and $f(y,g(y))$ is failed, though it can be unified with the infinite term $f(g(g(g(\cdots))),g(g(g(\cdots))))$.

There have been proposed several algorithms for unification with infinite terms [5, 7, 12]. The substantial difference is that expressions defining a binding environment can refer the environment itself recursively. Therefore, a looped infinite term such as $g(g(g(\cdots)))$ (the solution for $x = g(x)$) is permitted as a unifier. For instance, $g(x,f(y,h(x)),x)$ and $g(f(h(u),v),u,u)$ are unified to

$$g(f(h(f\cdots),h(f\cdots)),f(h(f\cdots),h(f\cdots))).$$

(i.e. The environment is $x = u = f(y,y),\ y = v = h(x)$.)

A looped infinite term can be represented by a cyclic finite graph as an internal form. Thus, the algorithm of infinite unification terminates as well as usual unification algorithms do. For details, refer [12].

Remark If two terms are unifiable under *unification with occur-check*, unifiable under *unification with infinite terms*. If two terms are unifiable under *unification with infinite terms*, unifiable under *unification without occur-check*. However, the converse will not be satisfied.

4.2 E-nonoverlapping property of a non-linear TRS

In this section, the decidable condition for E-nonoverlapping property is introduced. For the preparation, we introduce variations of overlapping conditions corresponding to variations of unifications. Let us first recall the definition of the overlapping condition.

Definition (again) A pair of reduction rules $\alpha_i \to \beta_i$ and $\alpha_j \to \beta_j$ is said to be *overlapping* iff there exists a context $C[\;]$, a nonvariable term M, and a substitution θ s.t. $\alpha_i \equiv C[M]$ and $\theta(\alpha_j) \equiv \theta(M)$ (i.e. α_j and M are unifiable).

In this definition, usual unification with occur-check is applied. Similarly, a pair of reduction rules is said to be *ω-overlapping* (resp. *strongly overlapping*) iff unification with infinite terms (resp. unification without occur-check) is applied instead of a usual unification with occur-check in the definition above.

Same as the definition of *nonoverlapping*, a TRS R is said to be *ω-nonoverlapping* (resp. *strongly nonoverlapping*) iff no pair of two rules in R are ω-overlapping (resp. strongly nonoverlapping) except trivial cases (i.e. $i = j \;\wedge\; C[\;] \equiv \Box$).

Theorem 2 If a TRS R is ω-nonoverlapping, then E-nonoverlapping.

The proof is found in **Appendix B**. The relation among these variations of nonoverlapping conditions is clarified as shown below.

Remark *strongly nonoverlapping*
$\Longrightarrow$ *ω-nonoverlapping*
$\Longrightarrow$ *E-nonoverlapping*
$\Longrightarrow$ *nonoverlapping* (See Figure 3)

Note that if R is left linear, all these nonoverlapping properties are equivalent.

The following two corollaries are direct consequences of the theorem.

Corollary 1 An ω-nonoverlapping TRS R is **UC**.

The assumption *ω-nonoverlapping* is weaker than *strongly nonoverlapping*, and the result **UC** is stronger than $\mathbf{UN}^{\to}$. Thus, **corollary 1** is a simple but more powerful result than the following classical theorem.

Figure 3: Relation among nonoverlapping properties.

Theorem [4] A TRS R is $\mathbf{UN}^{\to}$ if the following conditions are met :

- R is strongly nonoverlapping.
- R is compatible.

In fact, the theorem above shows that **Example 4** is $\mathbf{UN}^{\to}$. Further, **theorem 2** shows that the example is **UC**, though it is not **CR** (See Figure 4).

Example 4

$$R_4 \stackrel{\text{def}}{=} \left\{ \begin{array}{lcl} d(x,x) & \to & 0 \\ f(x) & \to & d(x, f(x)) \\ 1 & \to & f(1) \end{array} \right\}$$

The next corollary makes contrast with another classical result : *If a nonoverlapping TRS R is strongly normalizing* (i.e. $\forall x_0 \to x_1 \to x_2 \to \cdots \to x_i \to \cdots \exists n$ s.t. $x_n \in NF(R)$), *then* **CR** [10].

Corollary 2 If an *ω-nonoverlapping* TRS R is *weakly normalizing*, then **CR**.

The other approach to **CR**-related properties of a non-linear TRS is found in [16]. In [16], a nonoverlapping and nonlinear TRS is guaranteed to be **CR** by restricting its reductions in *call-by-value* strategy when *critical.* The main theorem is,

Theorem [16] If a membership conditional TRS R is nonoverlapping and *restricted-nonlinear*, then **CR**.

Figure 4: A **UC**, but not **CR** example R_4.

where a restricted-nonlinear membership conditional TRS reduces $\theta(\alpha_i)$ to $\theta(\beta_i)$ only when a substitution θ satisfies $\theta(x) \in NF(R)$ for all nonlinear variables in a rule $\alpha_i \to \beta_i \in R$.

5 Conclusion and future works

In this paper, a newly proposed *uniquely converging* (**UC**) property was investigated. **UC**, which is an intermediate property between **CR** and **UN**, was proposed in connection with the consistency of *continuous semantics*. Adding to it, a sufficient condition for **UC** was clarified. The statement is, *an ω-nonoverlapping TRS is* **UC** (irrespective of linearlity). The difference between *ω-nonoverlapping* and usual nonoverlapping is that *unification with infinite terms* was applied instead of usual unification with occur-check.

Equality among infinite objects (as shown in this paper) will be developed through following three stages:

- The definition of the equality (declarative semantics)
- The logical inference rules of the equality (underlying logic)
- The strategy to manipulate the equality (theorem prover)

This paper investigated only the first stage. The next may have two approaches. One approach is to give an adequate *conservative extension* based on final algebra, that is, add an adequate finite observation function. For instance in **Example 1**, $intseq(x)$ and $h(x)$ are distinguished with $car(cons(x, y)) \to x$, though still identifying $intseq(x)$ and $intseq'(x)$. In fact, it is proved that there exists a such conservative extension [15].

The other is to introduce induction rules to initial algebra, such as *fixed point induction* (in *LCF*) or *lazy induction* [3].

In either cases, further investigation is required in this area.

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Appendix

A Proof of theorem 1

In appendix A, we assume that R is an E-nonoverlapping TRS. Main proof techniques are various inductions based on various induction bases, such as a step $\#\Im_{M=N}$, a parallel step $\#_p\Im_{M=N}$, and a length $\Delta(M)$ of a term M.

Definition An equality-path $\Im_{M=N}$ is said to be a *parallel equality-path* iff any occurrences u, v $(u \neq v)$ where a reduction occurs in $\Im_{M=N}$ are disjoint (i.e. $u|v$).

A *parallel step* $\#_p\Im_{M=N}$ of an equality-path $\Im_{M=N}$ is a minimum number of a decomposition into parallel equality-pathes. That is, $\#_p\Im_{M=N}$ is formally defined as

$$min\left(\left\{ n \;\middle|\; \begin{array}{l} \forall i(\leq n)\ \exists M_i \text{ s.t. } M \equiv M_0,\ M_n \equiv N, \\ \text{and } \Im_{M=N} \equiv \Im_{M_0=M_1} \cdots \Im_{M_{n-1}=M_n} \\ \text{for some parallel equality-pathes} \\ \Im_{M_i=M_{i+1}} \end{array} \right\}\right)$$

Definition A *length* $\Delta(M)$ of the term M [8] is inductively defined as

$$\begin{cases} \Delta(x) & \stackrel{\text{def}}{=} 1 & \text{for any variable } x \\ \Delta(f(M_1, \cdots, M_n)) & \stackrel{\text{def}}{=} 1 + \sum_{i=1}^{n} \Delta(M_i) & \text{otherwise} \end{cases}$$

A.1 Step 1 : normalization lemma

Notation Substitutions θ, θ' are noted to be $\theta =_R \theta'$ (resp. $\theta \stackrel{*}{\rightarrow} \theta'$) iff $\theta(x) =_R \theta'(x)$ (resp. $\theta(x) \stackrel{*}{\rightarrow} \theta'(x)$) for any variable x.

Notation Let M, N be terms s.t. $M = N$, and $\Im_{M=N}$ be an equality-path. *Boundary* $\partial\Im_{M=N}$ is defined as

$$\partial\Im_{M=N} \stackrel{\text{def}}{=} Min\left(\left\{ u \;\middle|\; \begin{array}{l} \text{A reduction at an occur-} \\ \text{rence } u \text{ appears in } \Im_{M=N} \end{array} \right\}\right)$$

Elimination lemma Let $\alpha_i \rightarrow \beta_i \in R$. If $\theta(\alpha_i) =_R \theta'(\alpha_i)$ for some substitutions θ, θ', then $\theta =_R \theta'$. Further, $\theta =_R \theta'$ naturally induces $\theta(\beta_i) =_R \theta'(\beta_i)$ s.t. $\#_p\Im_{\theta(\beta_i)=\theta'(\beta_i)} \leq \#_p\Im_{\theta(\alpha_i)=\theta'(\alpha_i)}$.

Proof The latter part is obvious from the frontier part because any variables in β_i appears in α_i. The proof of the frontier part is due to the induction on a parallel step $\#_p\Im_{\theta(\alpha_i)=\theta'(\alpha_i)}$. The initial induction step $\#_p\Im_{\theta(\alpha_i)=\theta'(\alpha_i)} = 0$ is obvious.

If $\partial\Im_{\theta(\alpha_i)=\theta'(\alpha_i)} \cap Node^{\times}(\alpha_i) = \phi$, the lemma is obvious. Otherwise, from E-nonoverlapping property, there exist an occurrence $u \in \partial\Im_{\theta(\alpha_i)=\theta'(\alpha_i)}$, a rule $\alpha_j \to \beta_j \in R$, and substitutions σ, σ' s.t. $\Im_{\theta(\alpha_i)/u=\theta'(\alpha_i)/u}$ contains $\Im' \equiv \Im_{\sigma(\beta_j)\leftarrow\sigma(\alpha_j)} \cdot \Im_{\sigma(\alpha_j)=\sigma'(\alpha_j)} \cdot \Im_{\sigma'(\alpha_j)\leftarrow\sigma'(\beta_j)}$. Since $\#_p\Im' < \#_p\Im_{\theta(\alpha_i)/u=\theta'(\alpha_i)/u}$, $\Im'$ is shortened from the induction hypothesis. Thus, $\Im_{\theta(\alpha_i)=\theta'(\alpha_i)}$ is shortened, and again from the induction hypothesis, lemma is proved. ■

Definition An equality-path $\Im_{M=N}$ (resp. reduction-path $\Re_{M\overset{*}{\to}N}$) is said to be *normalized* iff the reduction at the occurrence u appears in $\Im_{M=N}$ (resp. $\Re_{M\overset{*}{\to}N}$) exactly once for $\forall u \in \partial\Im_{M=N}$ (resp. $\partial\Re_{M\overset{*}{\to}N}$).

E-nonoverlapping property and repeated applications of **elimination lemma** induce following **normalization lemma**.

Normalization lemma If $M = N$, there exists a normalized equation-path $\Im_{M=N}$.

A.2 Step 2 : proof for UN

UN-lemma An E-nonoverlapping TRS R is **UN**.

Proof Let $M, N \in NF(R)$ s.t. $M = N$ and the step of the equality-path $\#\Im_{M=N} = n$. We will prove $M \equiv N$ by induction on n.

The initial induction step, $M \equiv N$ for $n = 0$, is obvious.

Assume $M \equiv N$ holds for $\#\Im_{M=N} < n$ as induction hypothesis. Let $\exists M, N \in NF(R)$ s.t. $\#\Im_{M=N} = n$ and $M \not\equiv N$. From **normalization lemma**, there exists a normalized equation-path $\Im_{M=N}$. Without loss of generality, we can assume $\partial\Im_{M=N} = \{\epsilon\}$. Then, there exist M', N' s.t. $\Im_{M=N} = \Im_{M=M'} \cdot (M' \to N') \cdot \Im_{N'=N}$, $\epsilon \notin O_{inv}(\Im_{M=M'})$, and $\epsilon \notin O_{inv}(\Im_{N'=N})$.

Let $M' \to N'$ at ϵ by the rule $\alpha_i \to \beta_i$. If $\alpha_i \to \beta_i$ is a left linear reduction rule, *E-nonoverlapping* property and $\epsilon \in O_{inv}(\Im_{M=M'})$ implies $M \equiv \theta(\alpha_i)$ for some substitution θ. This contradicts to the assumption $M \in NF(R)$.

Then, $\alpha_i \to \beta_i$ must be a nonlinear rule. From *E-nonoverlapping* property and $M \in NF(R)$, there exist $u, v \in occur(\alpha_i, x)$ for some nonlinear variable x s.t. $u \neq v$, $M/u = M'/u$, $M/v = M'/v$, and $M'/u \equiv M'/v$.

Note that both $\Im_{M/u=M'/u}$ and $\Im_{M/v=M'/v}$ are subsequences of $\Im_{M=M'}$ (i.e. subsequences of $\Im_{M=N}$). Then, $\#(\Im_{M/u=M'/u} \cdot \Im_{M'/v=M/v}) < \#\Im_{M=N} = n$. From the facts $M/u \not\equiv M/v$ and $M/u, M/v \in NF(R)$, this contradicts to the induction hypothesis. ■

A.3 Step 3: proof for UC

The proof of **theorem 1** is due to the induction on the sum of the *lengths* of objective terms. Let us denote a *root function symbol* of a term M as $root(M)$.

Proof of theorem 1 The proof is due to the induction on $\Delta(\omega_R(P)) + \Delta(\omega_R(Q))$ where $P, Q \in \Theta_{\omega_R}(M) \overset{\text{def}}{=} \{\omega_R(N) \mid M =_R N\}$ for any term M[3].

The initial induction step $\Delta(\omega_R(P)), \Delta(\omega_R(Q)) = 1$ is the case that $P, Q \in V \cup \{\perp\}$. Since any left side of a rule α_i is not a variable from the definition of TRS, $V \subseteq NF(R)$. Thus, **UN lemma** implies that P, Q are cooperative.

Assume the theorem holds for $\Delta(\omega_R(P)) + \Delta(\omega_R(Q)) < n$ as induction hypothesis. Let $P, Q \in \Theta_{\omega_R}(M)$ s.t. $\Delta(\omega_R(P)) + \Delta(\omega_R(Q)) = n$ and $\omega_R(P), \omega_R(Q)$ are individual in $\Theta_{\omega_R}(M)$. Without loss of generality, we assume $\partial\Im_{P=Q} = \{\epsilon\}$ for some normalized equation-path $\Im_{P=Q}$ between P and Q. A pair P, Q is classified into following three cases.

[1] $P, Q \in NF(R)$
[2] $P \in NF(R)$, $Q \notin NF(R)$
(or $P \notin NF(R)$, $Q \in NF(R)$)
[3] $P, Q \notin NF(R)$

Case [1] leads the contradiction directly from **UN-lemma** and the fact $M \equiv \omega_R(M)$ for $\forall M \in NF(R)$.

In case [2], there exists terms P', Q' s.t. the sequence $\Im_{P=Q}$ is divided to $\Im_{P=P'} \cdot (P' \doteq Q') \cdot \Im_{Q'=Q}$ s.t. $\epsilon \notin \Im_{P=P'}$, $\epsilon \notin \Im_{Q'=Q}$, and the reduction between P' and Q' occurs at the root. If $P' \doteq Q'$ is realized as $P' \to Q'$ by the reduction rule $\alpha_i \to \beta_i$, then E-nonoverlapping property implies that P and P' have a same shape with α_i from the root[4]. Since $P \in NF(R)$, there must exist the distinct occurrences $u, v \in occur(\alpha_i, x)$ for some nonlinear variable x s.t. $P/u \not\equiv P/v$, $P/u = P/v$, and $P/u, P/v \in NF(R)$. This contradicts to **UN-lemma**.

If $P' \doteq Q'$ is realized as $Q' \to P'$ by the reduction rule $\alpha_i \to \beta_i$, then E-nonovelapping property implies that Q and Q' have a same shape with α_i from the root. If $\alpha_i \to \beta_i$ is a left linear rule, there must exists a substitution θ s.t. $Q \equiv \theta(\alpha_i)$ from E-nonoverlapping property. Then, $\omega_R(Q) \equiv \perp$. This contradicts to the assumption.

Thus, $\alpha_i \to \beta_i$ must be a nonlinear rule. There are again two cases; [2a] There exists a nonlinear variable x in α_i s.t. $\omega_R(Q/u)$ and $\omega_R(Q/v)$ for $u, v \in occur(\alpha_i, x)$ are individual in $T^\infty(F \cup \{\perp\}, X)$. (Thus, individual in $\Theta_{\omega_R}(M)$.) Since $Q/u =_R Q'/u \equiv Q'/v =_R Q/v$ and $\Delta(\omega_R(Q/u)) + \Delta(\omega_R(Q/v)) < \Delta(\omega_R(Q))$, this case contradicts to the induction hypothesis; [2b] For any nonlinear variable x in α_i, $\omega_R(Q/u)$ and $\omega_R(Q/v)$ are cooperative in $T^\infty(F \cup \{\perp\}, X)$.

[3] Note that $\Delta(\omega_R(P)), \Delta(\omega_R(Q)) < \infty$. because P, Q are deduced from a finite tree M in finite steps.

[4] i.e. $root(P/u) \equiv root(P'/u) \equiv root(\alpha_i/u)$ for $\forall u \in Node^{\times}(\alpha_i)$

(Thus, cooperative in $T(F,X)$.) Let $N_x \in T(F,X)$ s.t. $\omega_R(Q/u), \omega_R(Q/v) \sqsubseteq N_x$. Since $\bar{Q} \equiv Q[u \leftarrow N_x \mid \forall u \in occur(\alpha_i, x)$ for any nonlinear variable x in $\alpha_i]$ is a redex, and Q is obtained from $\bar{Q}$ by replacing subtrees with elements in $Cand_R^-$ (which correspond to $\perp$). Thus $Q \in Cand_R^-$, and this leads $\omega_R(Q) \equiv \perp$. This contradicts to the assumption.

In case [3], the proof is similar to the case [2a], [2b]. ■

B Proof of theorem 2

The proof is *boot-strapped* from simply nonoverlapping property. That is, E-nonoverlapping property is decomposed into $\langle E, n\rangle$-nonoverlapping property, which is valid only for *less-than-n-parallel-steps* equality-pathes, and stepwise refinement of **elimination lemma** and **normalization lemma** pull up it inductively. The main proof technique is the induction on parallel steps of equality-pathes.

Definition Let R be a TRS. A pair of reduction rules $\alpha_i \rightarrow \beta_i$ and $\alpha_j \rightarrow \beta_j$ is said to be $\langle E,n\rangle$-*overlapping* iff there exist a context $C[\,]$, a nonvariable term M, and a substitution θ s.t. $\alpha_i \equiv C[M]$, and $(\theta(\alpha_j) =_R \theta(M)) \wedge (\epsilon \in O_{inv}(\Im_{\theta(\alpha_j)=\theta(M)})) \wedge (\#_p \Im_{\theta(\alpha_j)=\theta(M)} \leq n)$.

A TRS R is said to be $\langle E,n\rangle$-*nonoverlapping* iff no pair of two rules in R are $\langle E,n\rangle$-overlapping except trivial cases (i.e. $i = j \ \wedge \ C[\,] \equiv \Box$).

Then, similar argument as in section 3.3.(1) leads a stepwise version of **elimination lemma** and **normalization lemma**.

Stepwise-elimination lemma Let a TRS R be $\langle E,n\rangle$-nonoverlapping, and $\alpha_i \rightarrow \beta_i \in R$. If $\theta(\alpha_i) =_R \theta'(\alpha_i)$ s.t. $\#_p \Im_{\theta(\alpha_i)=_R\theta'(\alpha_i)} \leq n$ for some substitutions θ, θ', then $\theta =_R \theta'$. Further, $\theta =_R \theta'$ naturally induces $\theta(\beta_i) =_R \theta'(\beta_i)$ s.t. $\#_p \Im_{\theta(\beta_i)=\theta'(\beta_i)} \leq \#_p \Im_{\theta(\alpha_i)=\theta'(\alpha_i)}$.

Stepwise-normalization lemma Let a TRS R be $\langle E,n\rangle$-nonoverlapping. If $M =_R N$ s.t. $\#_p \Im_{M=N} \leq n+1$, there exists a normalized equation-path $\Im'_{M=N}$ s.t. $\#_p \Im'_{M=N} \leq \#_p \Im_{M=N}$.

Proof of theorem 2 we will prove that R is $\langle E,n\rangle$-*nonoverlapping* by induction on n. The initial induction step is obvious because $\langle E,0\rangle$-nonoverlapping is equivalent to nonoverlapping, and the fact that ω-nonoverlapping implies nonoverlapping.

Assume R be $\langle E, n\text{-}1\rangle$-nonoverlapping as an induction hypothesis, and α_i and α_j be nontrivially $\langle E,n\rangle$-overlapping. That is, there exist a context $C[\,]$, a nonvariable term M, and a substitution θ s.t. $\alpha_i \equiv C[M]$, and $(\theta(\alpha_j) =_R \theta(M)) \wedge (\epsilon \in O_{inv}(\Im_{\theta(\alpha_j)=\theta(M)})) \wedge (\#_p \Im_{\theta(\alpha_j)=_R\theta(M)} \leq n)$.

From assumption, α_i and α_j are not *ω-overlapping* (except α_i overlaps with itself at the root). Thus, along the execution of the infinite unification algorithm on M (a nonvariable subterm of α_i) and α_j, there exist non-variable subterms P, P' of M or α_j s.t. some frontier $\{x\} = (P, P')$ failed. (That is, $root(P) \not\equiv root(P')$.)

There are two cases the frontier $\{x\} = (P, P')$ fails.

[1] P is a subterm of M, and P' is a subterm of α_j.
[2] P, P' are both subterms of M (or, α_j).

In case [1], there must exists a context $C'[\,]$ s.t.

- $C'[P]$ is a subterm of M.
- $C'[P']$ is a subterm of α_j.
- $\theta(C'[P]) =_R \theta(C'[P'])$ and $\Im_{\theta(C'[P])=\theta(C'[P'])}$ is a subsequence of $\Im_{\theta(M)=\theta(\alpha_j)}$.
  (i.e. $\#_p \Im_{\theta(C'[P])=\theta(C'[P'])} \leq \#_p \Im_{\theta(M)=\theta(\alpha_j)}$).

From **stepwise-normalization lemma**, we assume that $\Im_{\theta(C'[P])=\theta(C'[P'])}$ is normalized. Then, there exists an occurrence $u \in \partial\Im_{\theta(C'[P])=\theta(C'[P'])}$ and terms Q, Q' s.t.

- $u \preceq v$ where $\{v\} = occur(C'[\,], \Box)$
- $\Im_{\theta(C'[P])=\theta(C'[Q])} \equiv \Im_{\theta(C'[P])=Q} \cdot (Q \doteq Q') \cdot \Im_{Q'=\theta(C'[P'])}$ where $Q \doteq Q'$ is induced by a reduction at u.

Assume $Q \doteq Q'$ is $Q \rightarrow Q'$ by the reduction rule $\alpha_k \rightarrow \beta_k$. Then, α_i and α_k are E-overlapping in $\#_p \Im_{\theta(C'[P])=Q}$ parallel steps. If α_i and α_k are trivially E-overlapping (i.e. $i = k$ and overlaps at their roots), then $\Im_{\theta(M)=\theta(\alpha_j)}$ is shortened. This contradicts to the induction hypothesis. If not, this also contradicts to the induction hypothesis from $\#_p \Im_{\theta(C'[P])=Q} < \#_p \Im_{\theta(C'[P])=\theta(C'[P'])} (\leq \#_p \Im_{\theta(M)=\theta(\alpha_j)} = n)$.

In case [2], there must exist a nonlinear variable x in α_j (or, M) corresponding to the occurrences of P and P' in M (or, α_j). Then, either a pair of P and $\theta(x)$, or a pair of P' and $\theta(x)$ have different function symbols at their roots. Since both $\Im_{\theta(P)=\theta(x)}$ and $\Im_{\theta(P')=\theta(x)}$ are subsequences of $\Im_{\theta(M)=\theta(\alpha_j)}$, the case [2] is reduced to the case [1]. ■