Simple gap termination for term graph rewriting systems

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Abstract

This paper proves the Kruskal-type theorem with gap-condition á la Friedman on ω-trees (Main theorem 1 in section 2). As an application, it also proposes a termination criteria, named *simple gap termination* (Main theorem 2 in section 3), for term graph rewriting systems (on possibly cyclic terms), where the naive extension of simple termination (based on [Lav78]) for term graph rewriting systems does not work well.

1 Better-Quasi-Order

For infinite objects such as ω-trees, *Well-Quasi-Order* (WQO) does not close under the embedability construction. Instead, we need an extension of WQO, called *Better-Quasi-Order* (BQO). Note that (1) if $(Q, \leq)$ is a *well order* then $(Q, \leq)$ is a BQO, and if $(Q, \leq)$ is a BQO then $(Q, \leq)$ is a WQO, and (2) if Q is finite then $(Q, \leq)$ is BQO for any QO $\leq$[Lav78].

Definition 1.1 Let ω be the least countable ordinal (i.e., set of natural numbers). If $s, t \subseteq \omega$, then $s \leq t$ ($s < t$) means that s is a (proper) initial segment of t. Define $s \triangleleft t$ to hold if there is an $n > 0$ and $i_0 < \cdots < i_n < \omega$ s.t. for some $m < n$, $s = \{i_0, \cdots, i_m\}$ and $t = \{i_1, \cdots, i_n\}$. (Thus, e.g., $\{3\} \triangleleft \{5\}$, $\{3,5,6\} \triangleleft \{5,6,8,9\}$, $\{3,5,6\} \not\triangleleft \{5,6\}$.)

Definition 1.2 For an infinite set $X \subseteq \omega$, a *barrier* B on X is a set of finite sets of X s.t. $\phi \notin B$ and

1. for every infinite set $Y \subseteq \omega$ there is an $s \in B$ s.t. $s < Y$.
2. if $s, t \in B$ and $s \neq t$ then $s \not\subset t$.

Theorem 1.1 [1] If B is a barrier and $B = \cup_{i \leq n} B_i$ for some $n < \omega$, then some B_i contains a barrier (on $\cup_{b \in B_i} b$).

Definition 1.3 Let $\leq$ be a transitive binary relation on a set Q. Then,

- If $\leq$ is reflexive, R is called a *quasi-order* (QO).
- If $\leq$ is antisymmetric, R is called a *partial order* (or, simply *order*).
- If each pair of different elements in Q is comparable by $\leq$, $\leq$ is said to be *total.*

[1] Corollary 1.5 in [Lav78]. The proof is due to Galvin-Prikry. See Theorem 9.9 in [Sim85a].

A *strict* part of $\leq$ is $\leq - \geq$ and denoted as $<$. We also say a strict (quasi) order $<$ if it is a strict part of a (quasi) order $\leq$. When $\leq$ is a QO, we will sometimes use $\preceq$ (resp. $\prec$) instead of $\leq$ (resp. $<$), for clarity.

Definition 1.4 Let $\leq$ be a QO on Q. If B is a barrier, $f : B \to Q$ is *good* if there are $s, t \in B$ s.t. $s \lhd t$ and $f(s) \leq f(t)$, and f is *bad* otherwise. f is *perfect* if for all $s, t \in B$, if $s \lhd t$ then $f(s) \leq f(t)$. Q is *better-quasi-ordered* (bqo) if for every barrier B and every $f : B \to Q$, f is good.

Remark 1.1 If we restrict the BQO definition s.t. B runs only barriers of singleton sets (i.e., $B = \{1, 2, \cdots\}$), then we get the familiar well-quasi-order (WQO) definition.

A (possibly infinite) *tree* is a set of T on which a strict partial order $<_T$ is defined s.t. for every $t \in T$, $\{s \in T \mid s <_T t\}$ is well ordered under $<_T$. Thus $T = \cup_\alpha T_\alpha$ where α runs on ordinals and T_α, the α-th level of T, is the set of all $t \in T$ s.t. $\{s \mid s <_T t\}$ has type α. The *height* of T is the least α with $T_\alpha = \phi$. A *path* in T is a linearly ordered downward closed subset of T. If $x \in T$ (resp. a path P in T), let $S(x)$ (resp. $S(P)$) be the set of immediate successors of x (resp. P). A path is *maximal* in T if $S(P) = \phi$. Let $br_T(x)$ (or simply $br(x)$ if unambiguous) be $\{y \in T \mid x \leq_T y\}$, the *branch* above x. An ω-*tree* is a (possibly infinitely branching) tree of the height at most ω.

Definition 1.5 Let $\mathcal{T}$ be a set of trees which satisfies

1. For each $T \in \mathcal{T}$, T has a root (minimum element),
2. For each $T \in \mathcal{T}$, if P is a path in T with no largest element then $Card(S(P)) \leq 1$. A *Q-tree* $\mathcal{T}_Q$ is a pair (T, l) where $T \in \mathcal{T}$ and $l : T \to Q$.

If $T \in \mathcal{T}$, $s, t \in T$, there is a greatest lower bound of s and t in T, denoted by $s \wedge t$.

Definition 1.6 Let Q be a QO set and $(T_1, l_1), (T_2, l_2) \in \mathcal{T}_Q$. (T_1, l_1) is *embeddable* to (T_2, l_2) (and denoted $(T_1, l_1) \leq (T_2, l_2)$, or simply $T_1 \leq T_2$) if there exists $\psi : T_1 \to T_2$ s.t.

1. For $s, t \in T_1$, $\psi(s \wedge t) = \psi(s) \wedge \psi(t)$,
2. For $t \in T_1$, $l_1(t) \leq l_2(\psi(t))$.

Theorem 1.2 [Lav78, NW65] If Q is BQO, $\mathcal{M}_Q$ is BQO wrt the embedability $\leq$.

Remark 1.2 WQO is not enough for Kruskal-type theorem for infinite objects. For instance, consider $Q = \{(i, j) \mid i < j < \omega\}$ ordered by $(i, j) \leq (k, l)$ if and only if either $i = k$ *wedge* $j \leq k$ or $j < k$. Then Q is WQO, but a set Q^ω of infinite sequence on Q is not WQO, namely,

$$
\begin{array}{rcl}
f_1 & = & \langle (0,1), (1,2), (1,3), (1,4), \cdots \rangle, \\
f_2 & = & \langle (0,1), (1,2), (2,3), (2,4), \cdots \rangle, \\
\cdot & = & \cdot \\
\cdot & = & \cdot \\
f_i & = & \langle (0,1), \cdots, (i, i+1), (i, i+2), (i, i+3), \cdots \rangle, \\
\cdot & = & \cdot
\end{array}
$$

The main techniques to prove Kruskal-type theorems are (1) Ramsey-like theorem and (2) the existence of the *minimal bad sequence* (MBS). For (1), theorem 1.1 works. For (2), we first prepare some definitions (See [Lav78]).

Definition 1.7 Suppose Q is quasi-ordered by $\leq$. A *partial ranking* on Q is a well-founded (irreflexive) partial order $<'$ on Q s.t. $q <' r$ implies $q < r$. Let B, C be barriers. Then $B \sqsubseteq C$ if

1. $\cup C \subseteq \cup B$, and
2. for each $c \in C$ there is a $b \in B$ with $b \leq c$.

$B \sqsubset C$ if $B \sqsubseteq C$ and there are $b \in B$, $c \in C$ with $b < c$. For $f: B \to Q$, $g: C \to Q$ and a partial ranking $<'$ on Q, $f \sqsubseteq g$ ($f \sqsubset g$) wrt $<'$ if $B \sqsubseteq C$ ($B \sqsubset C$) and

1. $g(a) = f(a)$ for $a \in B \cap C$,
2. $g(c) <' f(b)$ for $b \in B$, $c \in C$ s.t. $b < c$.

Definition 1.8 Suppose $<'$ is a partial ranking on Q. For a barrier C, $g: C \to Q$ is *minimal bad* if g is bad and there is no bad h with $g \sqsubset h$.

Theorem 1.3 [2] Let Q be quasi-ordered by $\leq$, $<'$ a partial ranking on Q. Then for any bad f on Q there is minimal bad g s.t. $f \sqsubseteq g$.

Thus, the proof of Kruskal-type theorem on infinite objects is reduced to find some appropriate partial ranking $<'$.

2 Kruskal-type theorems with gap-condition on infinite trees

Kruskal's theorem with gap-condition for finite trees have been proposed for finite ordinals[Sim85b][3]. The aim of this section is to prove **main theorem 1**, which extends Kruskal's theorem with gap-condition to ω-trees. **Main theorem 1** is obtained as a corollary to the the stronger statement theorem 2.2). The scenario of its proof is similar to those that in [Lav78] and its extension is inspired by [Sim85b].

Definition 2.1 For $n < \omega$, let $\mathcal{M}_n$ be a set of ω-trees with labels in n $(= \{0, 1, \cdots, n-1\})$, and $(T_1, l_1), (T_2, l_2) \in \mathcal{M}_n$. $(T_1, l_1) \leq_G (T_2, l_2)$ if there exists $\psi: T_1 \to T_2$ s.t.

1. $T_1 \leq T_2$,
2. For each $t \in T_1$, $l_1(t) = l_2(\psi(t))$,
3. For $t \in T_1$, if there is $t' \in T_1$ s.t. $t \in S(t')$ then $l_2(s) \geq l_1(t)$ for each s s.t. $\psi(t') <_{T_2} s <_{T_2} \psi(t)$,
4. For the root t of T_1, $l_2(s) \geq l_1(t)$ for each s s.t. $s <_{T_2} \psi(t)$.

Theorem 2.1 [Sim85b] For $n < \omega$, let $T(n)$ be the set of all finite trees with labels less-than-equal n. Then $\leq_G$ is a WQO on the set $T(n)$.

Main theorem 1 *For $n < \omega$, let $\mathcal{M}_n$ be a set of ω-trees with labels in $n (= \{0, 1, \cdots, n-1\})$. Then $\mathcal{M}_n$ is BQO wrt $\leq_G$.*

To show the theorem, we will prove the slightly stronger statement.

[2] Theorem 1.9 in [Lav78], or equivalently theorem 9.17 in [Sim85a].

[3] There are two variants of its extensions for infinite ordinals[K$\check{8}$9, Gor90].

Definition 2.2 For $n < \omega$, let Q be a QO and $q: Q \to n (= \{0, 1, \cdots, n-1\})$. $\mathcal{M}_n(Q)$ is a set of ω-trees satisfying: for $(T, l) \in \mathcal{M}_n(Q)$, $l(t) \in n$ if $t \in T$ is not maximal wrt $<_T$ and $l(t) \in n \cup Q$ if $t \in T$ is maximal wrt $<_T$.

For $(T_1, l_1), (T_2, l_2) \in \mathcal{M}_n(Q)$, $(T_1, l_1) \leq_{\bar{G}} (T_2, l_2)$ if there exists $\psi: T_1 \to T_2$ s.t.

1. $T_1 \leq T_2$,
2. For each interior vertex $t \in T_1$, $\psi(t)$ is an interior vertex of T_2 and $l_1(t) = l_2(\psi(t))$,
3. For each end vertex $t \in T_1$, $\psi(t)$ is an end vertex of T_2 and either $l_1(t) = l_2(\psi(t)) \in n$ or $l_1(t) \leq l_2(\psi(t)) \in Q$.
4. For each interior vertex $t \in T_1$, $t' \in S(t)$ and $s \in T_2$ with $\psi(t) <_{T_2} s <_{T_2} \psi(t')$, $l_2(s) \geq l_1(\psi(t'))$ when $l_1(\psi(t')) \in n$ and $l_2(s) \geq q(l_1(\psi(t')))$ when $l_1(\psi(t')) \in Q$.
5. For the root t of T_1 and $s \in T_2$ with $s <_{T_2} \psi(t)$, $l_2(s) \geq l_1(\psi(t))$ when $l_1(\psi(t)) \in n$ and $l_2(s) \geq q(l_1(\psi(t)))$ when $l_1(\psi(t)) \in Q$.

We will denote $(T_1, l_1) \equiv (T_2, l_2)$ if $(T_1, l_1) \leq_{\bar{G}} (T_2, l_2)$ and $(T_1, l_1) \geq_{\bar{G}} (T_2, l_2)$

Theorem 2.2 Let $n < \omega$, Q be a BQO and $q: Q \to n$ $(= \{0, 1, \cdots, n-1\})$. Let $\mathcal{M}_n(Q)$ be the set of all ω-trees with labels in n for non-maximal verteces and with labels in $n \cup Q$ for maximal verteces. Then $\mathcal{M}_n(Q)$ is BQO wrt $\leq_{\bar{G}}$.

Definition 2.3 Let $n < \omega$. Let Q be a QO and $q: Q \to n$. $\mathcal{W}_n(Q), \mathcal{S}_n(Q), \mathcal{F}_n(Q) (\subseteq \mathcal{M}_n(Q))$ are defined to be:

1. $\mathcal{W}_n(Q)$ is a set of ω-words in $\mathcal{M}_n(Q)$.
2. $\mathcal{S}_n(Q)$ is a set of *scattered* ω-trees in $\mathcal{M}_n(Q)$. (i.e., for each $(S, l) \in \mathcal{S}_n(Q)$, $\eta \not\leq S$ where η is a complete binary ω-tree $(2)^\omega$.)
3. $\mathcal{F}_n(Q)$ is a set of *descensionally finite* trees. (i.e., For $(T, l) \in \mathcal{F}_n(Q)$, there is no infinite sequence $x_0 <_T x_1 <_T \cdots$ with $(br(x_0), l) >_{\bar{G}} (br(x_1), l) >_{\bar{G}} \cdots$.)

The scenario of the proof of theorem 2.2 consists of four steps: First, $\mathcal{W}_n(Q)$, which is a set of ω-words, is shown to be a BQO wrt $\leq_{\bar{G}}$ (theorem 2.3). Second, $\mathcal{S}_n(Q)$, which is a set of *scattered* ω-trees, is shown to be a BQO wrt $\leq_{\bar{G}}$ (theorem 2.4). During this step, the principle tool is a recursive definition of $\mathcal{S}_n(Q)$ which (a) starts with one-point or empty trees and (b) constructs the next stage using an element in $\mathcal{W}_n(Q)$ as a *spine*.

$(T, l) \in \mathcal{M}_n(Q)$ is a countable union of scattered ω-trees, i.e., $T = \cup_i S_i$ with $(S_i, l) \in \mathcal{S}_n(Q)$. Using this decomposition, thirdly $\mathcal{F}_n(Q)$, which is a set of *descensionally finite* ω-trees, is shown to be a BQO wrt $\leq_{\bar{G}}$ (theorem 2.5). Again using this decomposition, lastly $\mathcal{M}_n(Q) = \mathcal{F}_n(Q)$ is shown (theorem 2.6).

Theorem 2.3 Let $n < \omega$. For a barrier D, $g: D \to \mathcal{W}_n(Q)$ is bad wrt $\leq_{\bar{G}}$, then there is a barrier E and $g \sqsubseteq j$ s.t. $j: E \to Q$ is bad.

Proof Assume g is minimal bad wrt a partial ranking $<'$ on $\mathcal{W}_n(Q)$ where $J <' K$ if and only if $J \leq_G K$ and $dom(J) < dom(K)$. From theorem 1.1, we can assume $\forall d \in D$ s.t. either (1) $dom(g(d)) = 1$, (2) $dom(g(d)) < \omega$, or (3) $dom(g(d)) = \omega$.

For (1), there exists a barrier $E (\subseteq D)$ s.t. $g(e) \in Q$ for $e \in E$. By taking $j = g|_E$, theorem is proved.

For (2), we will prove by induction on n. Again by theorem 1.1, we can assume $\forall d \in D$ s.t. either (2-a) $g(d)$ does not contain 0, (2-b) the first element of $g(d)$ is 0, or (2-c) $g(d)$ contains 0 and the first element of $g(d)$ is not 0. For (2-a), by subtracting 1 from each label of $g(d)$, it is reduced to the induction hypothesis. For (2-b), let $g'(d)$ be obtained from $g(d)$ by taking the first element. Then, $g'(d)$ is bad and this contradicts to the minimal bad assumption of g. For (2-c), let $g(d) = (g_1(d), g_2(d))$. Since $g_1(d)$ and $g_2(d)$ are good from the minimal bad assumption of g, there is a barrier E s.t. $g_1(d)$ and $g_2(d)$ are perfect. This implies that $g(d)$ is good.

For (3), if $g(d_1) \not\leq_{\bar{G}} g(d_2)$ with $d_1 \lhd d_2$, there exists an initial segment J s.t. $J \not\leq_{\bar{G}} g(d_2)$. Let $h : D(2) \to (n)^{<\omega}$ by $h(d_1 \cup d_2) = J$. Then $g \sqsubset h$ and this contradicts to the minimal bad assumption on g. ∎

Definition 2.4 Let $T \in \mathcal{T}$, P a path in T, $z \in P$. Then let $\tilde{P}(z) = \{br(y) \mid y \in S(z) \text{ and } y \notin P\}$.

Lemma 2.1 (lemma 2.1 in [Lav78]) Let $n < \omega$ and Q be a QO. Let α be an ordinal and λ be a limit ordinal. Let

$$\begin{array}{lcl} \mathcal{S}^0(Q) & = & \{\text{the empty tree}\} \cup n \cup Q \\ \mathcal{S}^{\alpha+1}(Q) & = & \left\{ T \;\middle|\; \begin{array}{l} \text{there is a maximal path } P \in \mathcal{W}_n(Q) \text{ in } T \\ \text{s.t. } \tilde{P}(z) \subseteq \mathcal{S}^\alpha(Q) \text{ for all } z \in P \end{array} \right\} \\ \mathcal{S}^\lambda(Q) & = & \cup_{\alpha<\lambda} \mathcal{S}^\alpha. \end{array}$$

by regarding n, Q as one point trees. Then $\mathcal{S}_n(Q) = \cup_\alpha \mathcal{S}^\alpha(Q)$. We say $rank(T)$ for $T \in \mathcal{S}_n(Q)$ be the least α s.t. $T \in \mathcal{S}^\alpha(Q)$.

Theorem 2.4 Let $n < \omega$. For a barrier C, $g : C \to \mathcal{S}_n(Q)$ is bad wrt $\leq_{\bar{G}}$, then there is a barrier E and $g \sqsubseteq j$ s.t. $j : E \to Q$ is bad.

Proof Let a partial ranking $<'$ on $\mathcal{S}_n(Q)$ be $(T_1, l_1) <' (T_2, l_2)$ if $(T_1, l_1) \leq_{\bar{G}} (T_2, l_2)$ and $rank(T_1) < rank(T_2)$. Assume g is minimal bad wrt a partial ranking $<'$ on $\mathcal{S}_n(Q)$. From theorem 1.1, we can assume $\forall d \in C$ s.t. either (1) $card(g(d)) = 1$ or (2) $card(g(d)) > 1$. For (1), there exists a barrier $E(\subseteq C)$ s.t. $g(e) \in Q$ for $e \in E$. By taking $j = g|_E$, theorem is proved.

For (2), let $c \in C$. Let P_c be a maximal path in T_c where $g(c) = (T_c, l_c) \in \mathcal{S}_n(Q)$ s.t. for each $x \in P_c$ and each $T' \in \tilde{P}_c(x)$ $rank(T') < rank(T_c)$. Let $J_c : P_c \to \mathcal{W}_{n+1}(Q) \times \mathcal{P}(\mathcal{S}_n(Q))$ be defined by

$$J_c = (I_c(x), \tilde{P}_c(x))$$

where $I_c(x)$ is the sequence which is obtained by adding $n+1$ as the maximal element (wrt $<_{T_c}$) to the path from the root of T_c to x. By regarding J_c as a sequence, $J_c \leq J_d$ (embedability without gap-condition) implies $(T_c, l_c) \leq (T_d, l_d)$ for $c, d \in C$. From theorem 1.10 in [Lav78], if g is bad, there is a barrier D and $\bar{g} : D \to \mathcal{W}_{n+1}(Q) \times \mathcal{P}(\mathcal{S}_n(Q))$ s.t. $g \sqsubseteq \bar{g}$ and $\bar{g}$ is bad (by identifying an element as a sequence of the length 1). From theorem 2.3 and theorem 1.11 in [Lav78] (with $\leq_1$ on $\mathcal{P}(\mathcal{S}_n(Q))$, which is an one-to-one embedability on sets), there exists a barrier E and $j : E \to \mathcal{W}_{n+1}(Q) \times \mathcal{S}_n(Q)$ s.t. $D \subseteq E$ and j is bad. For $j(e) = (I_c(x), T')$ where $x \in P_e \subseteq T_c$ and each $T' \in \tilde{P}_c(x)$ for $c \sqsubseteq e$, let $j'(e)$ be a tree obtained by replacing the last element of $I_c(x)$ (whose label is $n+1$) with T'. $g \sqsubseteq j'$ and $rank(j'(e)) < rank(T_c)$ (since $rank(T') < rank(T_c)$ and adding a sequence to the root of T' does not change its rank). This contradicts to the minimal bad assumption of g. ∎

Adding (possibly infinite numbers of) finite trees to $(S, l) \in \mathcal{S}_n(Q)$ does not exceed the class of $\mathcal{S}_n(Q)$. Thus without loss of generality, for each $(T, l) \in \mathcal{M}_n(Q)$ we can assume the decomposition $T = \cup_i T_i$ with $(T_i, l) \in \mathcal{S}_n(Q)$ satisfies that if x is maximal wrt $<_{T_i}$ then either $br(x)$ does not contain 0 or $l(x) = 0$.

Definition 2.5 Let $(T,l) \in \mathcal{F}_n(Q)(\subseteq \mathcal{M}_n(Q))$ and $T = \cup_i T_i$ with $(T_i,l) \in \mathcal{S}_n(Q)$ s.t. if $x \in T_i$ is maximal wrt $<_{T_i}$ then either $br(x)$ does not contain 0 or $l(x) = 0$. If T does not contain a vertex labeled 0, $subt(T,l) \in \mathcal{F}_{n-1}(Q)$ is (T,l') where $l'(x) = l(x) - 1$ for each $x \in T$. With a fresh symbol Ω, let $Q^+ = Q \cup \{\Omega\}$ with $q(\Omega) = 0$ [4]. We denote $\mathcal{F}_n(Q)^{<(T,l)} = \{(U,m) \in \mathcal{F}_n(Q) \mid (U,m) <_{\bar{G}} (T,l)\}$.

Define $A_{(T,l)}(i) = (\bar{T}_i, \bar{l}) \in \mathcal{S}_{n+1}(Q^+ \cup \mathcal{F}_{n-1}(Q) \cup \mathcal{F}_n(Q)^{<(T,l)})$ where

1. If $x \in T_i$ is not maximal wrt $<_{T_i}$, then $\bar{l}(x) = l(x)$.
2. If $x \in T_i$ is maximal wrt $<_{T_i}$ and $(br(x), l)$ does not contain 0, then add a new vertex x^+ below x and set $\bar{l}(x) = n+1$, $\bar{l}(x^+) = subt(br(x), l)$.
3. If $x \in T_i$ is maximal wrt $<_{T_i}$, $l(x) = 0$ and $(br(x), l) <_{\bar{G}} (T,l)$, then $\bar{l}(x) = (br(x), l)$.
4. If $x \in T_i$ is maximal wrt $<_{T_i}$, $l(x) = 0$ and $(br(x), l) \equiv (T,l)$, then $\bar{l}(x) = \Omega$.

Define $A((T,l)) = \{A_{(T,l)}(i) \mid i < \omega\} \in \mathcal{P}(\mathcal{S}_{n+1}(Q^+ \cup \mathcal{F}_{n-1}(Q) \cup \mathcal{F}_n(Q)^{<(T,l)}))$. For $(T,l), (U,m) \in \mathcal{F}_n(Q)$, define $A((T,l)) \le A((U,m))$ if for each $A_{(T,l)}(i) \in A((T,l))$ there exists $A_{(U,m)}(j) \in A((U,m))$ s.t. $A_{(T,l)}(i) \le_{\bar{G}} A_{(U,m)}(j)$.

Lemma 2.2 For $(T,l), (U,m) \in \mathcal{F}_n(Q)$, $A((T,l)) \le A((U,m))$ implies $(T,l) \le_{\bar{G}} (U,m)$.

Proof We will construct an embedding $H : (T,l) \to (U,m)$ (with gap-condition) in ω steps. The induction hypothesis is:

If $x \in T_i$ is maximal wrt $<_{T_i}$, there is a 1-1 function J_i s.t.

1. if $(br(y), l)$ does not contain 0 then $(br(y), l) \le_{\bar{G}} (br(J_i(y)), m)$,
2. if $l(y) = 0$ and $(br(y), l) <_{\bar{G}} (T,l)$ then $m(J_i(y)) = 0$ and $(br(y), l) \le_{\bar{G}} (br(J_i(y)), m)$,
3. if $l(y) = 0$ and $(br(y), l) \equiv (T,l)$ then $m(J_i(y)) = 0$ and $(br(J_i(y)), m) \equiv (U,m)$.

Since $A((T,l)) \le A((U,m))$, there exists $A_{(U,m)}(j) \in A((U,m))$ s.t. $A_{(T,l)}(0) = (\bar{T}_0, \bar{l}) \le_{\bar{G}} A_{(U,m)}(j) = (\bar{U}_j, \bar{m})$. Then set H_0 by the embedding $T_0 \to U_j$.

Suppose that H_i has been defined, $y \in T_i$ is maximal. If either (1) $(br(y), l)$ does not contain 0 or (2) $l(y) = 0$ and $(br(y), l) <_{\bar{G}} (T,l)$ then $(br(y), l) \le_{\bar{G}} (br(J_i(y)), m)$. Thus extend H_i with an embedding of $br(y)$ into $br(J_i(y))$.

Suppose that (3) $l(y) = 0$ and $(br(y), l) \equiv \bar{G}(T,l)$ then there exists an embedding $L : (U,m) \to (br(J_i(y)), m)$. Since $A((T,l)) \le A((U,m))$, there exists $A_{(U,m)}(j) \in A((U,m))$ s.t. $A_{(T,l)}(i+1) = (\bar{T}_{i+1}, \bar{l}) \le_{\bar{G}} A_{(U,m)}(j) = (\bar{U}_j, \bar{m})$. Let $K : (T_{i+1}, l) \to (U_j, m) \subseteq (U,m)$ be an induced embedding. Thus extend H_i on $br(y) \cap T_{i+1}$ with LK. Since L isomorphically embeds (U,m) into $(br(J_i(y)), m)$, the induction hypothesis is satisfied to the next stage. ∎

Theorem 2.5 Let $n < \omega$. For a barrier B, $f : B \to \mathcal{F}_n(Q)$ is bad wrt $\le_{\bar{G}}$, then there is a barrier E and $f \sqsubseteq j$ s.t. $j : E \to Q$ is bad. Thus if Q is a BQO then $\mathcal{F}_n(Q)$ is a BQO (wrt $\le_{\bar{G}}$).

Proof We will prove by induction on n. For $n = 0$, $\le_{\bar{G}}$ and $\le$ (without gap-condition) are equivalent (see lemma 2 in theorem 2.4 of [Lav78]). Assume the theorem has been proved until $n-1$.

Define a partial ranking $<'$ by: $(U,m) <' (T,l)$ if and only if for some $x \in T$ $(U,m) = (br(x), l) <_{\bar{G}} (T,l)$. By theorem 1.3, we can assume $f : B \to \mathcal{F}_n(Q)$ is minimal bad. Let $f(b) = (T_b, l_b)$ for $b \in B$ and let $\bar{f}(b) = A((T_b, l_b))$. From lemma 2.2, $\bar{f}$ is bad. From lemma 1.3

[4] If Q is a BQO, Q^+ is also a BQO.

in [Lav78], there is a barrier $C \subseteq B(2)$ and an g defined on C s.t. for $c \in C$ ($c = b_1 \cup b_2$ where $b_1 \lhd b_2$ and $b_1, b_2 \in B$) $g(c) \in \bar{g}(b_1)$ and g is bad. Since $g(c) \in \mathcal{S}_{n+1}(Q^+ \cup \mathcal{F}_{n-1}(Q) \cup \mathcal{F}_n(Q)^{<(T_b,l_b)})$ and g is bad, from theorem 2.4 there is a barrier D with $C \sqsubset D$ and h defined on D s.t. $h(d) \in Q^+ \cup \mathcal{F}_{n-1}(Q) \cup \mathcal{F}_n(Q)^{<(T_b,l_b)}$ for $(b <)d \in D$ and h is bad. Since Q^+ and $\mathcal{F}_{n-1}(Q)$ are BQO, from theorem 1.1 there is a barrier $E \subseteq D$ and j defined on E s.t. $j(e) <' (T_b, l_b)$ for $(b <)e \in E$ and j is bad. Thus $g \sqsubset j$ and this is contradiction. ∎

Theorem 2.6 $\mathcal{M}_n(Q) = \mathcal{F}_n(Q)$.

We will prove theorem 2.6 by induction on n. For $n = 0$, $\leq$ and $\leq_{\bar{G}}$ are equivalent and this is shown by lemma 4 in theorem 2.4 in [Lav78]. Note that if $(T, l) \in \mathcal{M}_n(Q)$ does not contain 0, by induction hypothesis $subt(T, l) \in \mathcal{M}_{n-1}(Q) = \mathcal{F}_{n-1}(Q)$, and $(T, l) \in \mathcal{F}_n(Q)$.

Definition 2.6 Let $(T, l) \in \mathcal{M}_n(Q)$ and $T = \cup_i T_i$ with $(T_i, l) \in \mathcal{S}_n(Q)$ s.t. if $x \in T_i$ is maximal wrt $<_{T_i}$ then either $br(x)$ does not contain 0 or $l(x) = 0$. Let $Q^+ = Q \cup \{\Omega\}$ with $q(\Omega) = 0$.

Define $B_{(T,l)}(i) = (\bar{T}_i, \bar{l}) \in \mathcal{S}_{n+1}(Q^+ \cup \mathcal{F}_n(Q))$ where

1. If $x \in T_i$ is not maximal wrt $<_{T_i}$, then $\bar{l}(x) = l(x)$.
2. If $x \in T_i$ is maximal wrt $<_{T_i}$ and $(br(x), l)$ does not contain 0, then add a new vertex x^+ below x and set $\bar{l}(x) = n + 1$, $\bar{l}(x^+) = (br(x), l)$.
3. If $x \in T_i$ is maximal wrt $<_{T_i}$, $l(x) = 0$ and $br(x) \in \mathcal{F}_n(Q)$, then $\bar{l}(x) = (br(x), l)$.
4. If $x \in T_i$ is maximal wrt $<_{T_i}$, $l(x) = 0$ and $(br(x), l) \in \mathcal{M}_n(Q) - \mathcal{F}_n(Q)$, then $\bar{l}(x) = \Omega$.

Define $B((T, l)) = \{B_{(T,l)}(i) \mid i < \omega\} \in \mathcal{P}(\mathcal{S}_{n+1}(Q^+ \cup \mathcal{F}_n(Q)))$ For $(T, l), (U, m) \in \mathcal{M}_n(Q) - \mathcal{F}_n(Q)$, define $B((T, l)) \leq B((U, m))$ if for each $B_{(T,l)}(i) \in B((T, l))$ there exists $B_{(U,m)}(j) \in B((U, m))$ s.t. $B_{(T,l)}(i) \leq_{\bar{G}} B_{(U,m)}(j)$.

Lemma 2.3 Let $(T, l), (U, m) \in \mathcal{M}_n(Q) - \mathcal{F}_n(Q)$ s.t. $l(root(T)) = m(root(U)) = 0$. If $B((T, l)) \leq B((br(u), m))$ for each $u \in U$ s.t. $m(u) = 0$ and $(br(u, m)) \notin \mathcal{F}_n(Q)$, then $(T, l) \leq_{\bar{G}} (U, m)$.

Proof We will construct an embedding $I : (T, l) \to (U, m)$ (keeping gap-condition) in ω steps. The induction hypothesis is:

If $x \in T_i$ is maximal wrt $<_{T_i}$, there is a 1-1 function J_i s.t.

1. if $(br(y), l)$ does not contain 0 then $(br(J_i(y)), m)$ does not contain 0.
2. if $l(y) = 0$ and $(br(y), l) \in \mathcal{F}_n(Q)$ then $m(J_i(y)) = 0$ and $(br(J_i(y)), m) \in \mathcal{F}_n(Q)$,
3. if $l(y) = 0$ and $(br(y), l) \notin \mathcal{F}_n(Q)$ then $m(J_i(y)) = 0$ and $(br(J_i(y)), m) \notin \mathcal{F}_n(Q)$.

Since $B((T, l)) \leq B((U, m))$, there exists $B_{(U,m)}(j) \in B((U, m))$ s.t. $B_{(T,l)}(0) = (\bar{T}_0, \bar{l}) \leq_{\bar{G}} B_{(U,m)}(j) = (\bar{U}_j, \bar{m})$. Then set I_0 by the embedding $T_0 \to U_j$.

Suppose that I_i has been defined, $y \in T_i$ is maximal. If either (1) $br(y)$ does not contain 0 or (2) $l(y) = 0$ and $(br(y), l) \in \mathcal{F}_n(Q)$ then $(br(y), l) \leq_{\bar{G}} (br(J_i(y)), l)$. Thus extend I_i with an embedding of $br(y)$ into $br(J_i(y))$.

Suppose that (3) $l(y) = 0$ and $(br(y), l) \notin \mathcal{F}_n(Q)$, then from induction hypothesis $m(J_i(y)) = 0$ and $(br(J_i(y)), m) \notin \mathcal{F}_n(Q)$. Thus from the assumption, $B((T, l)) \leq B((br(J_i(y)), m))$ and there exists j s.t. $B_{(T,l)}(i + 1) \leq_{\bar{G}} B_{(br(J_i(y)),m)}(j)$ via an embedding K. Then I_i can be extended on $br(y) \cap T_{i+1}$ with K, and the induction hypothesis is preserved. ∎

Proof of induction step for theorem 2.6 Let $(T,l) \in \mathcal{M}_n(Q) - \mathcal{F}_n(Q)$ and $S = \{x \in T \mid l(x) = 0$ and $(br(x),l) \in \mathcal{M}_n(Q) - \mathcal{F}_n(Q)\}$. For each $s,t \in S$ s.t. $s <_T t$, $B((br(s),l)) \geq B((br(t),l))$ by an identity embedding.

If $(br(x),l)$ does not contain 0 then $(br(x),l) \in \mathcal{F}_n(Q)$. Thus S (wrt $<_T$) is an infinite tree of the height ω.

Since $B((T,l)) \in \mathcal{P}(\mathcal{S}_{n+1}(Q^+ \cup \mathcal{F}_n(Q)))$, $\{B((U,m)) \mid (U,m) \in \mathcal{M}_n(Q) - \mathcal{F}_n(Q)\}$ is a BQO, thus well-founded. Then there exists $s \in S$ s.t. for each $t \in S$ with $s <_T t$ $B((br(s),l)) \not> B((br(t),l))$ (thus $B((br(s),l)) \equiv B((br(t),l)))$. From lemma 2.3, $(br(s),l) \leq_{\bar{G}} (br(t),l)$. But since $(br(s),l) \in \mathcal{M}_n(Q) - \mathcal{F}_n(Q)$, from definition there must be an infinite sequence $s = s_0 <_T s_1 <_T \cdots$ s.t. $(br(s_i),l) >_{\bar{G}} (br(s_{i+1},l)$ for each i. This is contradiction. ∎

3 Simple gap termination for term graph rewriting systems

A reduction $\rightarrow$ is *terminating* if there is no infinite sequence s.t. $s_1 \rightarrow s_2 \rightarrow \cdots$. Simple termination [Der82] is the frequently used criteria for a term rewriting system. For a *term graph rewriting system* (TGRS) on possibly cyclic term graphs, the naive extension of simple termination based on Kruskal-type theorem on infinite trees [NW65, Lav78] does not work well. Let $R = \{a(a(b(x))) \rightarrow a(b(x))\}$. Then R is terminating. R rewrites a term graph $y : a(a(b(y)))$ to $y : a(b(y))$, but $unfold(y : a(a(b(y))) \geq unfold(y : a(b(y)))$ and $unfold(y : a(a(b(y))) \leq unfold(y : a(b(a(b(y))))) = unfold(y : (a(b(y))))$, because only fairness of occurrences of a, b on each path relates to $\leq$.

Definition 3.1 [JKdV94] A *term graph* s is a finite directed graph satisfying:

1. s has a root.
2. each vertex of s has a label (function symbol) which has a fixed arity.

An ω-term obtained by unfolding s is denoted $unfold(s)$. A *term graph rewriting system* (TGRS, for short) R is a finite set of *rewrite rules* $l \rightarrow r$ which are pairs of acyclic term graphs l, r s.t. l is not a variable and $V(l) \supseteq V(r)$.

Roughly speaking, reduction relation $\rightarrow$ is defined similar to those which of a term rewriting system, except that a TGRS regards a variable as an address. For precise definition, please refer [JKdV94, AK94]. We will consider reduction $\rightarrow$ of a TGRS on possibly cyclic term graphs[5].

Main theorem 2 *Let $R = \{l \rightarrow r\}$ be a TGRS. Assume that a set of function symbols is totally ordered. If there is a QO $\leq$ on ground term graphs s.t.*

1. *$s > t$ implies $C[s] > S[t]$ for each context $C[\ ]$.*
2. *$C[s] \geq s$ where each function symbol f on a path from the root of $C[s]$ to the root of s satisfies $f \geq root(s)$.*
3. *For each ground term graphs s, t, $s \xrightarrow[l \rightarrow r]{\lambda} t$ (i.e., reduction at the root by the rule $l \rightarrow r$) implies $s > t$.*

[5]The definition of reduction of TGRS on a cyclic term graph requires some unfolding mechanism for a term graph. For instance, when the rule $a(x) \rightarrow x$ is applied on a term graph $y : a(y)$, [JKdV94] asserts $y : a(y)$ as the result of the reduction. This requires some unfolding mechanism by default - otherwise, the result would be $y : y$. However this mechanism is not explicitly defined in literatures. Our termination criteria - *simple gap termination* (for a TRS see [Oga94]), on which unfolding does not effect - is a safer choice.

4. $s > t$ implies $unfold(s) \neq unfold(t)$.

Then R is terminating.

Proof Define a QO $\leq_{uf}$ on ω-trees by: $unfold(s) \leq_{uf} unfold(t)$ if $s \leq t$. From (4), $s > t$ implies $unfold(s) >_{uf} unfold(t)$. From (2), $C[unfold(s)] \geq_{uf} unfold(s)$ if each function symbol f on a path from the root of $C[unfold(s)]$ to the root of s satisfies $f \geq root(unfold(s))$. Since $unfold(s)$ has repeated patterns (produced by cycles in s) except for its downward-closed finite subset, thus $C[unfold(s)] \geq_{uf} unfold(s)$ and transitivity implies $\leq_{uf} \subseteq \leq_G$ on ω-trees obtained by unfolding finite term graphs.

Suppose there exists an infinite reduction sequence $s_1 \rightarrow s_2 \rightarrow \cdots$. Without loss of generality, we can assume that each s_i is a ground term graph. Thus from (1),(3), $s_1 > s_2 > \cdots$ and $unfold(s_1) >_{uf} unfold(s_2) >_{uf} \cdots$. However, from the main theorem 1 there exists i, j s.t. $i < j$ and $unfold(s_i) \leq_G unfold(s_j)$. This is contradiction. ∎

Then $y : a(a(b(y))) \rightarrow y : a(b(y))$ for $R = \{a(a(b(x))) \rightarrow a(b(x))\}$, and $unfold(y : a(a(b(y))) >_G unfold(y : a(b(y)))$ with $a > b$.

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