Lexicographic Combination of Reduction Pairs

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Abstract. We present a simple criterion for combining reduction pairs lexicographically. The criterion is applicable to arbitrary classes of reduction pairs, such as the polynomial interpretation, the matrix interpretation, and the Knuth–Bendix order. In addition, we investigate a variant of the matrix interpretation where the lexicographic order is employed instead of the usual component-wise order. Effectiveness is demonstrated by experiments and examples, including Touzet's Hydra Battle.

Keywords: Term rewriting · Termination · Dependency pairs.

1 Introduction

Lexicographic combination is a powerful method to combine termination measures into a more complex one. To illustrate it, consider the term rewrite system:

$$1\colon\quad \mathsf{f}(\mathsf{f}(x)) \to \mathsf{g}(\mathsf{g}(\mathsf{f}(x))) \qquad\qquad 2\colon\quad \mathsf{g}(\mathsf{g}(x)) \to x$$

For example, it has the next rewrite sequence:

$$\mathsf{f}(\mathsf{f}(\mathsf{f}(x))) \xrightarrow{1} \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{f}(x)))) \xrightarrow{2} \mathsf{f}(\mathsf{f}(x)) \xrightarrow{1} \mathsf{g}(\mathsf{g}(\mathsf{f}(x))) \xrightarrow{2} \mathsf{f}(x)$$

If the numbers of occurrences of f and g are measured, the sequence turns into the descending sequence with respect to the lexicographic order $>^{\mathsf{lex}}$:

$$(3,0) >^{\mathsf{lex}} (2,2) >^{\mathsf{lex}} (2,0) >^{\mathsf{lex}} (1,2) >^{\mathsf{lex}} (1,0)$$

In this manner, lexicographic combination can be used for showing termination of the rewrite system. More formally speaking, the measure can be expressed as the lexicographic combination of two linear polynomial interpretations [20], namely $\mathcal{A}$ and $\mathcal{B}$ defined by $\mathsf{f}_{\mathcal{A}}(x) = x + 1$, $\mathsf{g}_{\mathcal{A}}(x) = x$, $\mathsf{f}_{\mathcal{B}}(x) = x + 1$, and $\mathsf{g}_{\mathcal{B}}(x) = x + 1$. Here $\mathcal{A}$ counts the number of occurrences of f, while $\mathcal{B}$ counts that of g.

There has been a long line of research concerning termination analysis of term rewrite systems. Among others, the dependency pair framework [1,10,11,13,14] is a powerful method for automated termination analysis of term rewrite systems. Decreasing measures for the method are typically given in the form of *reduction pairs* consisting of preorders and well-founded orders on terms. Despite the powerfulness of lexicographic combination, it has been unexplored in the context of the dependency pair framework. This is because, in general, lexicographic combinations of reduction pairs are not reduction pairs. To overcome this, we give a simple criterion for a combination to be a reduction pair.

Actually, the previous example implicitly uses the folklore that lexicographic combinations of *monotone* reduction pairs are (monotone) reduction pairs, see [3,9,12]. Our main result can be conceived as an extension of it. For example, the following term rewrite system

$$\mathsf{s}(x) + y \to \mathsf{p}(\mathsf{s}(x)) + \mathsf{s}(y) \qquad\qquad \mathsf{p}(\mathsf{s}(x)) \to x$$

encodes addition in a tricky way, using the successor symbol s and the predecessor symbol p. The dependency pair framework tells us that the termination is established if one can find a reduction pair $(\geqslant, >)$ that fulfills the following set of constraints.

$$\begin{aligned} \mathsf{s}(x) + y &\geqslant \mathsf{p}(\mathsf{s}(x)) + \mathsf{s}(y) & \qquad \mathsf{p}(\mathsf{s}(x)) &\geqslant x \\ \mathsf{s}(x) +^\sharp y &> \mathsf{p}(\mathsf{s}(x)) +^\sharp \mathsf{s}(y) & \qquad \mathsf{s}(x) +^\sharp y &> \mathsf{p}^\sharp(\mathsf{s}(x)) \end{aligned}$$

Here, $+^\sharp$ and $\mathsf{p}^\sharp$ are fresh symbols introduced by the method. To do this, one can use the lexicographic combination of the following linear polynomial interpretations $\mathcal{A}$ and $\mathcal{B}$:

$$\begin{array}{lllll} \mathsf{s}_\mathcal{A}(x) = x + 1 & \mathsf{p}_\mathcal{A}(x) = x & \mathsf{p}^\sharp_\mathcal{A}(x) = 0 & x +_\mathcal{A} y = x & x +^\sharp_\mathcal{A} y = x \\ \mathsf{s}_\mathcal{B}(x) = x + 1 & \mathsf{p}_\mathcal{B}(x) = 0 & \mathsf{p}^\sharp_\mathcal{B}(x) = 0 & x +_\mathcal{B} y = x & x +^\sharp_\mathcal{B} y = x \end{array}$$

The resulting lexicographic combination satisfies the set of constraints. For example, $\mathsf{s}(x) +^\sharp y > \mathsf{p}(\mathsf{s}(x)) +^\sharp \mathsf{s}(x)$ is verified by the lexicographic comparison $x + 1 \geqslant x + 1$ and $x + 1 > 0$, corresponding to $\mathcal{A}$ and $\mathcal{B}$, respectively. A problem here is that, in contrast to the previous example, the algebras $\mathcal{A}$ and $\mathcal{B}$ are not monotone, as $\mathsf{p}^\sharp_\mathcal{A}(x) = \mathsf{p}^\sharp_\mathcal{B}(x) = 0$ disregards the argument x. Although the folklore does not apply, our result can justify that the resulting combination is indeed a reduction pair, which facilitates a successful termination proof.

The polynomial interpretation [20], the matrix interpretation [8], and the Knuth–Bendix order [19] are typical methods to construct reduction pairs. We also show how to use these methods with lexicographic combination, demonstrating it with the term rewrite system of the Battle of Hercules and Hydra [18] due to Touzet [25]. As a by-product, we obtain a variant of the matrix interpretation with the lexicographic order instead of the standard component-wise order. As remarked in [8,22], this has been an open question.

The structure of the paper. After recalling some preliminaries in Section 2, a criterion for lexicographic combination and termination proofs with it are presented in Section 3. The termination proof of the Hydra Battle is discussed in Section 4. Then, as a generalization of lexicographic combination of the linear polynomial interpretation, Section 5 studies a variant of the matrix interpretation with the lexicographic order. Experimental data and related work are discussed in Section 6 and Section 7.

2 Preliminaries

Throughout the paper, we assume familiarity with term rewriting [2,23]. In this section, we briefly recall notions and notations for term rewriting and termination analysis based on the dependency pair framework.

Let $\mathcal{F}$ be a signature and $\mathcal{V}$ an infinite set of variables. The set of terms built from $\mathcal{F}$ and $\mathcal{V}$ is denoted by $\mathcal{T}(\mathcal{F},\mathcal{V})$, or simply by $\mathcal{T}$. A mapping $\sigma : \mathcal{V} \to \mathcal{T}$ is called a *substitution* if $\sigma(x) \neq x$ for only finitely many variables x. Given a term t, the term obtained by replacing each variable occurrence x in t with $\sigma(x)$ is denoted by $t\sigma$. Let $\Box$ be a fresh constant symbol. A *context* is a term with exactly one occurrence of $\Box$. The term obtained by replacing $\Box$ in a context C with a term t is denoted by $C[t]$. We write $s \trianglerighteq t$ if $s = C[t]$ for some context C, and moreover we write $s \triangleright t$ if $s \trianglerighteq t$ and $s \neq t$. Let $\hookrightarrow$ be a relation on terms. The relation $\hookrightarrow$ is said to be *closed under substitutions* if $s\sigma \hookrightarrow t\sigma$ whenever $s \hookrightarrow t$ and σ is a substitution. Similarly, $\hookrightarrow$ is said to be *closed under contexts* if $C[s] \hookrightarrow C[t]$ whenever $t \hookrightarrow u$.

A *rewrite rule* $\ell \to r$ is a pair (ℓ, r) of terms such that ℓ is not a variable and every variable occurring in r also occurs in ℓ. A set of rewrite rules is called a *term rewrite system* (TRS). The *rewrite step* $\to_\mathcal{R}$ of a TRS $\mathcal{R}$ is defined as follows: $s \to_\mathcal{R} t$ if $s = C[\ell\sigma]$ and $t = C[r\sigma]$ for some rewrite rule $\ell \to r \in \mathcal{R}$, context C, and substitution σ. When $C = \Box$, it is written as $s \xrightarrow{\epsilon}_\mathcal{R} t$. Given a term t and a symbol x, we write $|t|_x$ for the number of occurrences of x in t. A term rewrite system $\mathcal{R}$ is *non-duplicating* if $|\ell|_x \geqslant |r|_x$ for all $\ell \to r \in \mathcal{R}$ and variables x. A term t is *terminating* with respect to a relation $\hookrightarrow$ on terms if there is no infinite sequence $t \hookrightarrow t_1 \hookrightarrow t_2 \hookrightarrow \cdots$ starting from t. *Termination* of the relation $\hookrightarrow$ is defined as absence of non-terminating terms. A TRS $\mathcal{R}$ is *terminating* if $\to_\mathcal{R}$ is. Given TRSs $\mathcal{R}$ and $\mathcal{S}$, the relation $\to_\mathcal{S}^* \cdot \to_\mathcal{R} \cdot \to_\mathcal{S}^*$ is denoted by $\to_{\mathcal{R}/\mathcal{S}}$. We say that $\mathcal{R}$ is *relatively terminating* with respect to $\mathcal{S}$, or $\mathcal{R}/\mathcal{S}$ is terminating, if $\to_{\mathcal{R}/\mathcal{S}}$ is terminating.

We recall the dependency pair framework [1,12,13,14,15]. Given a non-variable term $t = f(t_1,\ldots,t_n)$, we write $t^\sharp$ for $f^\sharp(t_1,\ldots,t_n)$, and $\mathrm{root}(t)$ for f. Here $f^\sharp$ is a fresh n-ary function symbol corresponding to f. The set $\{t^\sharp \mid t \in \mathcal{T}(\mathcal{F},\mathcal{V}) \setminus \mathcal{V}\}$ is denoted by $\mathcal{T}^\sharp$. Let $\mathcal{R}$ be a TRS. We define $\mathcal{D}_\mathcal{R}$ as $\{f \mid f(\ell_1,\ldots,\ell_n) \to r \in \mathcal{R}\}$. Elements in $\mathcal{D}_\mathcal{R}$ are called *defined symbols*. The TRS $\mathsf{DP}(\mathcal{R})$ is defined as follows:

$$\mathsf{DP}(\mathcal{R}) = \{\ell^\sharp \to t^\sharp \mid \ell \to r \in \mathcal{R},\ r \trianglerighteq t,\ \mathrm{root}(t) \in \mathcal{D}_\mathcal{R},\text{ and } \ell \ntrianglerighteq t\}$$

Rules in $\mathsf{DP}(\mathcal{R})$ are called *dependency pairs*. *Dependency pair problems* are pairs $(\mathcal{P},\mathcal{R})$ of TRSs with $\mathcal{P} \subseteq \mathcal{T}^\sharp \times \mathcal{T}^\sharp$ and $\mathcal{R} \subseteq \mathcal{T} \times \mathcal{T}$. A dependency pair problem $(\mathcal{P},\mathcal{R})$ is *finite* if there exists no infinite sequence of the form $s_1 \to_\mathcal{R}^* t_1 \xrightarrow{\epsilon}_\mathcal{P} s_2 \to_\mathcal{R}^* t_2 \xrightarrow{\epsilon}_\mathcal{P} \cdots$, where each s_i is terminating with respect to $\mathcal{R}$.

Theorem 1 ([11])**.** *A TRS $\mathcal{R}$ is terminating if and only if $(\mathsf{DP}(\mathcal{R}),\mathcal{R})$ is finite.*

A pair of a preorder $\geqslant$ and a strict order $>$ on the same set is an *order pair* if the inclusion $\geqslant \cdot > \cdot \geqslant\ \subseteq\ >$ holds. An order pair is *well-founded* if $>$ is well-founded. An order pair $(\geqslant, >)$ on terms is *stable* if $\geqslant$ and $>$ are closed under substitutions. A well-founded stable order pair is a *reduction pair* if $\geqslant$ is closed under contexts. If in addition $>$ is closed under contexts, $(\geqslant, >)$ is called a *monotone* reduction pair.

Theorem 2. *Let $(\geqslant, >)$ be a reduction pair. A dependency pair problem $(\mathcal{P}, \mathcal{R})$ with $\mathcal{P} \cup \mathcal{R} \subseteq \geqslant$ is finite if and only if $(\mathcal{P} \setminus >, \mathcal{R})$ is finite.*

Thus, a TRS $\mathcal{R}$ is terminating if $\mathsf{DP}(\mathcal{R}) \subseteq >$ and $\mathcal{R} \subseteq \geqslant$ for some reduction pair $(\geqslant, >)$. This simple criterion can also be used for showing relative termination. We say that a TRS $\mathcal{R}$ *dominates* a TRS $\mathcal{S}$ if r has no defined symbols of $\mathcal{R}$ for all rules $\ell \to r \in \mathcal{S}$.

Theorem 3 **([15]).** *Let $\mathcal{R}$ and $\mathcal{S}$ be TRSs such that $\mathcal{R}$ dominates $\mathcal{S}$ and $\mathcal{S}$ is non-duplicating. If there exists a reduction pair $(\geqslant, >)$ with $\mathsf{DP}(\mathcal{R}) \subseteq >$ and $\mathcal{R} \cup \mathcal{S} \subseteq \geqslant$ then $\mathcal{R}/\mathcal{S}$ is terminating.*

Let $\mathcal{F}$ be a signature. An *$\mathcal{F}$-algebra* $\mathcal{A}$ is a pair $(A, \{f_\mathcal{A}\}_{f \in \mathcal{F}})$ where A is a non-empty set, called the *carrier* of $\mathcal{A}$, and each $f_\mathcal{A}$ is an n-ary function on A, called the interpretation of an n-ary function symbol f. Let $\mathcal{A}$ be an $\mathcal{F}$-algebra. A function α from $\mathcal{V}$ to A is called an *assignment* for $\mathcal{A}$. It is extended to the homomorphism $[\alpha]_\mathcal{A} : \mathcal{T}(\mathcal{F}, \mathcal{V}) \to A$ as follows:

$$[\alpha]_\mathcal{A}(x) = \begin{cases} \alpha(x) & \text{if } x \text{ is a variable} \\ f_\mathcal{A}([\alpha]_\mathcal{A}(t_1), \ldots, [\alpha]_\mathcal{A}(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

Assume that $\mathcal{A}$ is equipped with an order pair $(\geqslant, >)$ on A. The algebra $\mathcal{A}$ is *well-founded* if $>$ is well-founded, and *weakly monotone* if $f_\mathcal{A}(a_1, \ldots, a_i, \ldots, a_n) \geqslant f_\mathcal{A}(a_1, \ldots, b, \ldots, a_n)$ whenever $f \in \mathcal{F}$ and $a_i \geqslant b$. We write $s \geqslant_\mathcal{A} t$ and $s >_\mathcal{A} t$ if $[\alpha]_\mathcal{A}(s) \geqslant [\alpha]_\mathcal{A}(t)$ and $[\alpha]_\mathcal{A}(s) > [\alpha]_\mathcal{A}(t)$ hold for all assignments α, respectively. It is known that $(\geqslant_\mathcal{A}, >_\mathcal{A})$ is a reduction pair if $\mathcal{A}$ is weakly monotone and well-founded.

The *matrix interpretation* [8] provides a semantic method to construct reduction pairs. The carrier of a matrix interpretation $\mathcal{A}$ is the set of vectors $\boldsymbol{x}$ of natural numbers of a fixed dimension $d > 0$. Vectors are ordered by the component-wise order pair $(\geqslant, >)$ defined as follows: $(x_1, \ldots, x_d)^T \geqslant (y_1, \ldots, y_d)^T$ if $x_i \geqslant y_i$ for all $i \in \{1, \ldots, d\}$; if in addition $x_1 > y_1$ then $(x_1, \ldots, x_d)^T > (y_1, \ldots, y_d)^T$. We write $\boldsymbol{e}_i$ for the unit vector with $(\boldsymbol{e}_i)_i = 1$ and $(\boldsymbol{e}_i)_j = 0$ for other j, and $(A)_{i,j}$ or simply $A_{i,j}$ for the entry at the i-th row and the j-th column. The interpretation $f_\mathcal{A}$ of each n-ary function symbol f is of the form

$$f_\mathcal{A}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = A_1\boldsymbol{x}_1 + \ldots + A_n\boldsymbol{x}_n + \boldsymbol{a}$$

where $A_1, \ldots, A_n$ are $d \times d$ matrices of natural numbers and $\boldsymbol{a} \in \mathbb{N}^d$. The pair $(\geqslant_\mathcal{A}, >_\mathcal{A})$ is a reduction pair. The special class of the matrix interpretation with $d = 1$ is called the *linear polynomial interpretation*.

The *Knuth–Bendix order* [19] is another way to construct reduction pairs. A *weight function* is a pair of a positive integer w_0 and a function w from function symbols to natural numbers with $w(c) \geqslant w_0$ for constants c. The *weight* $w(t)$ of a term t is defined inductively: $w(x) = w_0$ for variables x, and $w(t) = w(f) + \sum_i w(t_i)$ for $t = f(t_1, \ldots, t_n)$. We write $f^n(t)$ for the n-times application of a unary function symbol f to a term t. Let (w_0, w) be a weight function and $\succ$ a precedence, that is, a strict order on function symbols. The Knuth–Bendix order $>_{\mathsf{kbo}}$ is inductively defined as follows: $s >_{\mathsf{kbo}} t$ if $|s|_x \geqslant |t|_x$ for all variables x, and

1. $w(s) > w(t)$, or
2. $w(s) = w(t)$ and one of the following conditions holds.
   a. $s = f^n(t)$ for some $n > 0$ and t is a variable.
   b. $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and either
      i. $f \succ g$ or
      ii. $f = g$ and there is an i such that $s_i >_{\mathsf{kbo}} t_i$ and $s_j = t_j$ for all $j < i$.

The weight function (w_0, w) is *admissible* for $\succ$, if $f \succ g$ for other function symbols g whenever f is a unary function symbol with $w(f) = 0$. If $\succ$ is well-founded and (w_0, w) is admissible for $\succ$, then $(\geqslant_{\mathsf{kbo}}, >_{\mathsf{kbo}})$ is a monotone reduction pair, where $\geqslant_{\mathsf{kbo}}$ stands for the reflexive closure of $>_{\mathsf{kbo}}$. (We note that in that case $>_{\mathsf{kbo}}$ is a so-called reduction order.)

Argument filtering [1] is a popular transformation for building reduction pairs with the Knuth–Bendix order. As the name suggests, this transformation filters out arguments from a given term. Formally, an *argument filter* π is a mapping that associates each n-ary function symbol f to an integer i or a list $[i_1, \ldots, i_m]$ of integers over $\{1, \ldots, n\}$. The *argument filtering* $\hat{\pi}$ is defined as follows:

$$\hat{\pi}(t) = \begin{cases} t & \text{if } t \text{ is a variable} \\ \hat{\pi}(t_i) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } \pi(f) = i \\ f(\hat{\pi}(t_{i_1}), \ldots, \hat{\pi}(t_{i_m})) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } \pi(f) = [i_1, \ldots, i_m] \end{cases}$$

Note that arities of function symbols may change by applying $\hat{\pi}$. Given a binary relation R on terms, we write $s\ R^{\pi}\ t$ if $\hat{\pi}(s)\ R\ \hat{\pi}(t)$.

Proposition 4. *Let $(\geqslant, >)$ be a reduction pair. Then $(\geqslant^{\pi}, >^{\pi})$ is a reduction pair for every argument filter π.*

3 Combinability Criterion

Lexicographic combination is a well-known method to turn two order pairs into a single order pair which is capable of capturing a more complicated termination measure.

Definition 5. *Let $(\geqslant_1, >_1)$ and $(\geqslant_2, >_2)$ be order pairs on a set X. The* lexicographic combination *$(\geqslant_{12}, >_{12})$ is the pair of relations on X defined as follows:*

- *$x \geqslant_{12} y$ if $x >_1 y$, or both $x \geqslant_1 y$ and $x \geqslant_2 y$.*
- *$x >_{12} y$ if $x >_1 y$, or both $x \geqslant_1 y$ and $x >_2 y$.*

Lexicographic combinations of reduction pairs satisfy all conditions to be reduction pairs, except for closure under contexts of preorders.

Example 6. Consider the reduction pairs $(\geqslant_{\mathcal{A}}, >_{\mathcal{A}})$ and $(\geqslant_{\mathcal{B}}, >_{\mathcal{B}})$ induced by the following linear polynomial interpretations $\mathcal{A}$ and $\mathcal{B}$:

$$\begin{array}{lll} \mathsf{f}_{\mathcal{A}}(x) = 0 & \mathsf{a}_{\mathcal{A}} = 1 & \mathsf{b}_{\mathcal{A}} = 0 \\ \mathsf{f}_{\mathcal{B}}(x) = x & \mathsf{a}_{\mathcal{B}} = 0 & \mathsf{b}_{\mathcal{B}} = 1 \end{array}$$

Although $\mathsf{a} \geqslant_{\mathcal{AB}} \mathsf{b}$ follows from $\mathsf{a} >_{\mathcal{A}} \mathsf{b}$, the desired inequality $\mathsf{f}(\mathsf{a}) \geqslant_{\mathcal{AB}} \mathsf{f}(\mathsf{b})$ for closure under contexts does not hold. Even worse, the flipped inequality $\mathsf{f}(\mathsf{b}) >_{\mathcal{AB}} \mathsf{f}(\mathsf{a})$ follows from $\mathsf{f}(\mathsf{b}) \geqslant_{\mathcal{A}} \mathsf{f}(\mathsf{a})$ and $\mathsf{f}(\mathsf{b}) >_{\mathcal{B}} \mathsf{f}(\mathsf{a})$.

As noted in [28], the problem in Example 6 is that, in the first component $\mathsf{f}_{\mathcal{A}}(x) = 0$ does not respect the argument x whereas $\mathsf{f}_{\mathcal{B}}(x) = x$ does. This observation suggests the following definitions.

Definition 7. *Let $(\geqslant, >)$ be an order pair on terms. Let f be an n-ary function symbol. The i-th argument position of f is* $>$-monotone *(or monotone with respect to $>$) if*

$$t_i > u \implies f(t_1, \ldots, t_i, \ldots, t_n) > f(t_1, \ldots, u, \ldots, t_n)$$

for all terms $t_1, \ldots, t_n, u$. Similarly, the position is $\geqslant$-invariant *if*

$$f(t_1, \ldots, t_i, \ldots, t_n) \geqslant f(t_1, \ldots, u, \ldots, t_n)$$

for all terms $t_1, \ldots, t_n, u$. We say that order pairs $(\geqslant_1, >_1)$ and $(\geqslant_2, >_2)$ on terms are combinable *if $(\geqslant_1, >_1)$ is normal and every argument position of any function symbol is $>_1$-monotone or $\geqslant_2$-invariant. Here an order pair $(\geqslant, >)$ is* normal *if $> \subseteq \geqslant$.*

Notice that, if we consider the equivalence relation $\sim$ induced by $\geqslant$, invariance is equivalent to $f(t_1, \ldots, t_i, \ldots, t_n) \sim f(t_1, \ldots, u, \ldots, t_n)$ for all $t_1, \ldots, t_n, u$. This justifies the name of the property.

Recall the form of linear polynomial interpretations $\mathcal{A}$:

$$f_{\mathcal{A}}(x_1, \ldots, x_n) = a_0 + a_1 x_1 + \cdots + a_n x_n$$

Trivially, the i-th argument position of f is monotone with respect to $>_{\mathcal{A}}$ if $a_i > 0$, and invariant with respect to $\geqslant_{\mathcal{A}}$ otherwise. Moreover, $(\geqslant_{\mathcal{A}}, >_{\mathcal{A}})$ is normal.

Example 8 (continued from Example 6). The first argument position of f is neither monotone with respect to $>_{\mathcal{A}}$ nor invariant with respect to $\geqslant_{\mathcal{B}}$. Thus, the reduction orders $(\geqslant_{\mathcal{A}}, >_{\mathcal{A}})$ and $(\geqslant_{\mathcal{B}}, >_{\mathcal{B}})$ are *not* combinable.

We have similar facts for matrix interpretations $\mathcal{A}$: let the interpretation of an n-ary function symbol f be $f_{\mathcal{A}}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = A_1 \boldsymbol{x}_1 + \cdots + A_n \boldsymbol{x}_n + \boldsymbol{a}$. The i-th argument position of f is monotone if $(A_i)_{1,1} > 0$, and invariant if $A_i = O$, see [8]. Again, $(\geqslant_{\mathcal{A}}, >_{\mathcal{A}})$ is normal.

Lemma 9. *Let* $(\geqslant_{12}, >_{12})$ *be the lexicographic combination of combinable reduction pairs* $(\geqslant_1, >_1)$ *and* $(\geqslant_2, >_2)$*. Then* $\geqslant_{12}$ *is closed under contexts.*

Proof. It is sufficient to show the monotonicity of $\geqslant_{12}$, meaning that if $t_i \geqslant_{12} u$ then $C[t_i] \geqslant_{12} C[u]$ for a context C of the form $C = f(t_1, \ldots, \Box, \ldots, t_n)$. Suppose $t_i \geqslant_{12} u$. If it follows from $t_i \geqslant_1 u$ and $t_i \geqslant_2 u$, then $C[t_i] \geqslant_{12} C[u]$ follows from the assumptions that $\geqslant_1$ and $\geqslant_2$ are closed under contexts. Otherwise, $t_i \geqslant_{12} u$ follows from $t_i >_1 u$. According to the combinability, the i-th argument position of f is monotone in $>_1$ or invariant in $\geqslant_2$. In the former case $C[t_i] >_1 C[u]$ follows from $t_i >_1 u$. In the latter case, $t_i \geqslant_1 u$ from the normality, and therefore $C[t_i] \geqslant_1 C[u]$ from the closure under contexts. In addition, the invariance yields $C[t_i] \geqslant_2 C[u]$. Hence, in either case $C[t_i] \geqslant_{12} C[u]$ is concluded. ◻

Theorem 10. *Lexicographic combinations of combinable reduction pairs are reduction pairs.* ◻

We demonstrate a termination proof based on lexicographic combination.

Example 11. Recall the TRS $\mathcal{R}$ in the introduction:

$$1\colon\ \mathsf{s}(x) + y \to \mathsf{p}(\mathsf{s}(x)) + \mathsf{s}(y) \qquad\qquad 2\colon\ \mathsf{p}(\mathsf{s}(x)) \to x$$

The set $\mathsf{DP}(\mathcal{R})$ consists of two dependency pairs:

$$3\colon\ \mathsf{s}(x) +^\sharp y \to \mathsf{p}(\mathsf{s}(x)) +^\sharp \mathsf{s}(y) \qquad\qquad 4\colon\ \mathsf{s}(x) +^\sharp y \to \mathsf{p}^\sharp(\mathsf{s}(x))$$

Let $\mathcal{A}$ and $\mathcal{B}$ be the linear interpretations given by:

$$\begin{array}{lllll} \mathsf{s}_\mathcal{A}(\underline{x}) = x + 1 & \mathsf{p}_\mathcal{A}(\underline{x}) = x & \mathsf{p}^\sharp_\mathcal{A}(x) = 0 & \underline{x} +_\mathcal{A} y = x & \underline{x} +^\sharp_\mathcal{A} y = x \\ \mathsf{s}_\mathcal{B}(x) = x + 1 & \mathsf{p}_\mathcal{B}(\overline{x}) = 0 & \mathsf{p}^\sharp_\mathcal{B}(\overline{x}) = 0 & x +_\mathcal{B} \overline{y} = x & x +^\sharp_\mathcal{B} \overline{y} = x \end{array}$$

Here monotone positions in $\mathcal{A}$ and invariant positions in $\mathcal{B}$ are indicated by underlining and overlining, respectively. The induced reduction pairs are combinable, so let $(\geqslant_{\mathcal{AB}}, >_{\mathcal{AB}})$ be their lexicographic combination. Then $\geqslant_{\mathcal{AB}}$ orients the rules in $\mathcal{R}$ while $>_{\mathcal{AB}}$ orients those in $\mathsf{DP}(\mathcal{R})$:

$$\begin{array}{llll} 1\colon\ x + 1 \geqslant_\mathcal{A} x + 1 & 2\colon\ x + 1 >_\mathcal{A} x & 3\colon\ x + 1 \geqslant_\mathcal{A} x + 1 & 4\colon\ x + 1 >_\mathcal{A} 0 \\ 1\colon\ x + 1 \geqslant_\mathcal{B} 0 & & 3\colon\ x + 1 >_\mathcal{B} 0 & \end{array}$$

Hence, the termination of $\mathcal{R}$ is concluded by Theorem 1.

Remark 12. The normality requirement cannot be dropped from the combinability criterion (Theorem 10), see Appendix A. However, normality is a mild requirement, in the sense that typical reduction pairs like Knuth–Bendix order [19], the recursive path order [6,16], the matrix interpretation [8], and the polynomial interpretation [20] are all normal reduction pairs. Furthermore, lexicographic combinations of normal reduction pairs are normal.

If monotone and invariant positions of combined reduction pairs are identified, one can combine multiple reduction pairs by successively applying Theorem 10. Let $(\geqslant_{12}, >_{12})$ be the lexicographic combination of combinable reduction pairs $(\geqslant_1, >_1)$ and $(\geqslant_2, >_2)$.

Lemma 13. *If the i-th argument position of a function symbol is $>_2$-monotone and $>_2$ is non-empty then the position is $>_1$-monotone. Similarly, if the i-th argument position of a function symbol is $\geqslant_1$-invariant and $>_1$ is non-empty then the position is $\geqslant_2$-invariant.*

Proof. We only show the first claim because the second is shown in a similar way. Due to the non-emptiness of $>_2$, there are terms t and u with $t >_2 u$. By monotonicity $f(t, \ldots, t, \ldots, t) >_2 f(t, \ldots, u, \ldots, t)$ is obtained. So the i-th argument position of f is not $\geqslant_2$-invariant, and therefore the claim follows from the combinability. □

Theorem 14. *If the i-th argument of a function symbol is $>_2$-monotone and $>_2$ is non-empty, then it is $>_{12}$-monotone too. Similarly, If the i-th argument of a function symbol is $\geqslant_1$-invariant and $>_1$ is non-empty, then it is $\geqslant_{12}$-invariant too.*

Proof. Again, we only show the first claim. Let $t = f(t_1, \ldots, t_i, \ldots, t_n)$, $u = f(t_1, \ldots, u', \ldots, t_n)$, and $t_i >_{12} u'$. If $t_i >_{12} u'$ is due to $t_i >_1 u'$ then Lemma 13 yields $t >_1 u$, from which $t >_{12} u$ follows. Otherwise, $t_i \geqslant_1 u'$ and $t_i >_2 u'$. As $\geqslant_1$ is closed under contexts and the i-th position is $>_2$-monotone, the inequalities $t \geqslant_1 u$ and $t >_2 u$ follow. Hence, $t >_{12} u$ is concluded. □

The non-emptiness condition in Theorem 14 cannot be dropped, see Appendix A. In practice, this is not a problem, because a usual reduction pair (including those mentioned in Remark 12) can be extended to another one $(\geqslant, >)$ so that satisfies $\mathsf{c}_1 > \mathsf{c}_2$ for fresh constant symbols c_1 and c_2.

Example 15 (continued from Example 11). We show the termination of the extended system $\mathcal{R}' = \mathcal{R} \cup \{\mathsf{0} + y \to y\}$. The set $\mathsf{DP}(\mathcal{R}')$ coincides with $\mathsf{DP}(\mathcal{R})$. The linear interpretation $\mathcal{C}$ with

$$\mathsf{0}_{\mathcal{C}} = 1 \quad \mathsf{s}_{\mathcal{C}}(\underline{x}) = x \quad \mathsf{p}_{\mathcal{C}}(\underline{x}) = x \quad \mathsf{p}^\sharp_{\mathcal{C}}(x) = 0 \quad \underline{x} +_{\mathcal{C}} \underline{y} = x + y \quad \underline{x} +^\sharp_{\mathcal{C}} y = x$$

satisfies $\mathsf{0} + y >_{\mathcal{C}} y$ and $\mathcal{R}' \cup \mathsf{DP}(\mathcal{R}) \subseteq \geqslant_{\mathcal{C}}$. Therefore, $\mathcal{R} \subseteq \geqslant_{\mathcal{CAB}}$ and $\mathsf{DP}(\mathcal{R}) \subseteq >_{\mathcal{CAB}}$ are obtained if we extend $\mathcal{A}$ and $\mathcal{B}$ with $\mathsf{0}_{\mathcal{A}} = 0$ and $\mathsf{0}_{\mathcal{B}} = 0$. The combinability can be verified by Theorem 14. Hence, $\mathcal{R}$ is terminating.

Note that combinability is associative in the following sense: Suppose that the pairs of $(\geqslant_1, >_1)$ and $(\geqslant_2, >_2)$ and of $(\geqslant_2, >_2)$ and $(\geqslant_3, >_3)$ are combinable. Then, combinability of $(\geqslant_{12}, >_{12})$ and $(\geqslant_3, >_3)$ is equivalent to that of $(\geqslant_1, >_1)$ and $(\geqslant_{23}, >_{23})$, provided that $>_1$, $>_2$, and $>_3$ are non-empty. This is a consequence of Lemma 13.

4 Termination of the Battle of Hercules and Hydra

In this short section, we demonstrate a termination proof based on heterogeneous combination of reduction pairs. Among others, we pick up Touzet's TRS encoding of the Battle of Hercules and Hydra [25]; see also [7] for its backgrounds. Touzet's rewrite system $\mathcal{H}$ consists of the following eleven rules:

$$
\begin{array}{rlrl}
1: & ⟦\, \circ\, x \to \circ\, ⟦\, x & 6: & \mathsf{H}(\mathsf{0}, x) \to \circ\, x \\
2: & \bullet\, ⟦\, x \to ⟦\, \bullet \bullet\, x & 7: & \bullet\, \mathsf{H}(\mathsf{H}(\mathsf{0}, y), z) \to \mathsf{c}^1(y, z) \\
3: & \circ\, x \to \bullet\, ⟦\, x & 8: & \bullet\, \mathsf{H}(\mathsf{H}(\mathsf{H}(\mathsf{0}, x), y), z) \to \mathsf{c}^2(x, y, z) \\
4: & \bullet\, x \to x & 9: & \bullet\, \mathsf{c}^1(x, y) \to \mathsf{c}^1(x, \mathsf{H}(x, y)) \\
5: & \mathsf{c}^1(y, z) \to \circ\, z & 10: & \bullet\, \mathsf{c}^2(x, y, z) \to \mathsf{c}^2(x, \mathsf{H}(x, y), z) \\
 & & 11: & \mathsf{c}^2(x, y, z) \to \circ\, \mathsf{H}(y, z)
\end{array}
$$

Symbols $⟦$, $\circ$, and $\bullet$ are unary function symbols and their parentheses are omitted here.

While the original termination proof of $\mathcal{H}$ uses a sophisticated algebra on $\mathbb{O} \times \mathbb{N} \times \mathbb{N}$, our proof employs a combination of the ordinal interpretation on $\mathbb{O}$ and the Knuth–Bendix order. Here $\mathbb{O}$ is the set of all ordinal numbers below ϵ_0. In order to ease the proof, we introduce an easy corollary of Theorems 1 and 2. Below, we write $\mathcal{E}\mathsf{mb}$ for the TRS consisting of the *embedding rule* $f(x_1, \ldots, x_n) \to x_i$ for all n-ary function symbols f and $1 \leqslant i \leqslant n$.

Corollary 16. *A TRS $\mathcal{R}$ is terminating if $\mathcal{R} \subseteq >$ and $\mathcal{E}\mathsf{mb} \subseteq \geqslant$ for some reduction pair $(\geqslant, >)$.*

Actually, we use the Knuth–Bendix order together with argument filtering. For verifying combinability, we need to identify monotone and invariant positions. Consider an arbitrary argument filter π. We say that the argument position i of a function symbol f is *regarded* by π if $\pi(f) = i$ or $\pi(f) = [\ldots, i, \ldots]$. Let $(\geqslant, >)$ be a monotone reduction pair. Not surprisingly, the i-th argument position is $>^\pi$-monotone if it is regarded by π, and $\geqslant^\pi$-invariant otherwise.

We are ready to prove the termination of $\mathcal{H}$. Consider the algebra $\mathcal{O}$ on $\mathbb{O}$ with interpretations:

$$
\begin{array}{llll}
\mathsf{0}_{\mathcal{O}} = 0 & \mathsf{c}^1_{\mathcal{O}}(\underline{x}, y) = y + \omega^{x+1} & \circ_{\mathcal{O}}(\underline{x}) = x & ⟦_{\mathcal{O}}(\underline{x}) = x \\
\mathsf{H}_{\mathcal{O}}(\underline{x}, \underline{y}) = \omega^x \oplus y & \mathsf{c}^2_{\mathcal{O}}(\underline{x}, y, \underline{z}) = z \oplus \omega^{y + \omega^{x+1}} & \bullet_{\mathcal{O}}(\underline{x}) = x &
\end{array}
$$

Here $\oplus$ stands for natural addition, and monotone positions are indicated by underlining. The above interpretation is borrowed from the original proof. We have $\{5, 6, 11\} \cup \mathcal{E}\mathsf{mb} \subseteq >_{\mathcal{O}}$ and $\{1{-}4, 7{-}10\} \subseteq \geqslant_{\mathcal{O}}$, see [25, Lemma 10]. By taking the argument filter π with $\pi(\mathsf{0}) = \pi(\mathsf{H}) = \pi(\mathsf{c}^1) = \pi(\mathsf{c}^2) = [\,]$, and $\pi(\circ) = \pi(\bullet) = \pi(⟦) = [1]$ the rules in the latter group are simplified as follows:

$$
\begin{array}{llll}
1:\ ⟦\, \circ\, x \to \circ\, ⟦\, x & 3:\ \circ\, x \to \bullet\, ⟦\, x & 7:\ \bullet\, \mathsf{H} \to \mathsf{c}^1 & 9:\ \bullet\, \mathsf{c}^1 \to \mathsf{c}^1 \\
2:\ \bullet\, ⟦\, x \to ⟦\, \bullet \bullet\, x & 4:\ \bullet\, x \to x & 8:\ \bullet\, \mathsf{H} \to \mathsf{c}^2 & 10:\ \bullet\, \mathsf{c}^2 \to \mathsf{c}^2
\end{array}
$$

Consider the Knuth–Bendix order $>_{\mathsf{kbo}}$ induced from the precedence $\bullet \succ [\,] \succ \circ \succ \mathsf{c}^1 \succ \mathsf{c}^2$ and the admissible weight function (w_0, w) with:

$$w(\circ) = 2 \qquad w_0 = w(\mathsf{0}) = w(\mathsf{H}) = w(\mathsf{c}^1) = w(\mathsf{c}^2) = w([\,]) = 1 \qquad w(\bullet) = 0$$

It is not difficult to see that the Knuth–Bendix order orients the above simplified rules strictly, and thus, $\{1{-}4, 7{-}10\} \subseteq >^{\pi}_{\mathsf{kbo}}$. As non-monotone argument positions with respect to $>_{\mathcal{O}}$ are filtered out by π, the reduction pairs $(\geqslant_{\mathcal{O}}, >_{\mathcal{O}})$ and $(\geqslant^{\pi}_{\mathsf{kbo}}, >^{\pi}_{\mathsf{kbo}})$ are combinable. Hence, the termination is concluded by Corollary 16.

5 Echelon-Form Matrix Interpretation

Linear polynomial interpretations combined lexicographically can be regarded as a variant of the matrix interpretation which uses the lexicographic order pairs $(\geqslant^{\mathsf{lex}}, >^{\mathsf{lex}})$. Here $(x_1, \ldots, x_n)^T >^{\mathsf{lex}} (y_1, \ldots, y_n)^T$ if there exists an index $1 \leqslant i \leqslant n$ such that $x_i > y_i$ and $x_j = y_j$ for all $j < i$. The relation $\geqslant^{\mathsf{lex}}$ is the reflexive closure of $>^{\mathsf{lex}}$.

Example 17 (continued from Example 15). The three linear interpretations $\mathcal{C}$, $\mathcal{A}$, and $\mathcal{B}$ can be combined into the single algebra $\mathcal{M}$ on $\mathbb{N}^3$ with the interpretations:

$$\mathsf{s}_{\mathcal{M}}(\boldsymbol{x}) = \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad \mathsf{p}_{\mathcal{M}}(\boldsymbol{x}) = \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&0 \end{pmatrix} \boldsymbol{x} \qquad \mathsf{0}_{\mathcal{M}} = \boldsymbol{e}_1$$

$$\boldsymbol{x} +_{\mathcal{M}} \boldsymbol{y} = \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 1&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix} \boldsymbol{y} \quad \boldsymbol{x} +^{\sharp}_{\mathcal{M}} \boldsymbol{y} = \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix} \boldsymbol{x} \quad \mathsf{p}^{\sharp}_{\mathcal{M}}(\boldsymbol{x}) = \boldsymbol{0}$$

For instance, $\mathsf{p}_{\mathcal{M}}(\boldsymbol{x}) = (x_1, x_2, 0) = (\mathsf{p}_{\mathcal{C}}(x_1), \mathsf{p}_{\mathcal{A}}(x_2), \mathsf{p}_{\mathcal{B}}(x_3))$ holds for $\boldsymbol{x} = (x_1, x_2, x_3)^T$. If we equip $\geqslant^{\mathsf{lex}}$ with $\mathcal{M}$, the reduction pair $(\geqslant_{\mathcal{M}}, >_{\mathcal{M}})$ coincides with the lexicographic combination employed in Example 15.

Let $\mathcal{M}$ be a matrix interpretation equipped with the lexicographic order. Although $\mathcal{M}$ is well-founded, it cannot afford weak monotonicity for free, which is a relevant property for $(\geqslant_{\mathcal{M}}, >_{\mathcal{M}})$ to be a reduction pair. Below, we show that weak monotonicity is characterized by *(column) echelon form* matrices.

Let A be an $m \times n$ matrix. The matrix A is in *(column) echelon form* if the following property holds for all $1 \leqslant i \leqslant m$ and $2 \leqslant j \leqslant n$: If $A_{k,\,j-1} = 0$ for all $k < i$ then $A_{i,j} = 0$. For example, the 2×2 and 3×3 echelon-form matrices are classified as follows:

$$\begin{pmatrix} 0&0 \\ *&0 \end{pmatrix} \quad \begin{pmatrix} +&0 \\ *&* \end{pmatrix} \quad \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ *&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0 \\ +&0&0 \\ *&*&0 \end{pmatrix} \quad \begin{pmatrix} +&0&0 \\ *&0&0 \\ *&*&0 \end{pmatrix} \quad \begin{pmatrix} +&0&0 \\ *&+&0 \\ *&*&* \end{pmatrix}$$

Here $+$ stands for an arbitrary positive natural number, and $*$ for an arbitrary non-negative integer. The matrices $\left(\begin{smallmatrix} 0&0 \\ 0&1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0&0&0 \\ 0&0&0 \\ 0&1&0 \end{smallmatrix}\right)$ are not in echelon form.

These examples tell us that echelon matrices are lower-triangular matrices and the converse does not hold. The former fact can be easily confirmed from the definition.

Lemma 18. *Let A be an $m \times n$ matrix in echelon form. Then $A\boldsymbol{x} \geqslant^{\mathsf{lex}} A\boldsymbol{y}$ whenever $\boldsymbol{x} \geqslant^{\mathsf{lex}} \boldsymbol{y}$.*

Proof. We proceed by mathematical induction on m. If $n = 1$ then the claim is trivial, so assume $n > 1$. We further analyze the first column of A. If $A_{i,1} = 0$ for all $1 \leqslant i < m$ (which is in particular true when $m = 1$), from the echelon formedness it holds that $A_{i,j} = 0$ except for $i = m$ and $j = 1$, and therefore

$$A\boldsymbol{x} = \begin{pmatrix} \mathbf{0} \\ A_{m,1}x_1 \end{pmatrix} \geqslant^{\mathsf{lex}} \begin{pmatrix} \mathbf{0} \\ A_{m,1}y_1 \end{pmatrix} = A\boldsymbol{y}.$$

Otherwise, by calculation, we have:

$$A = \begin{pmatrix} \mathbf{0} & O \\ A_{i,1} & \mathbf{0}^T \\ \boldsymbol{a} & A' \end{pmatrix} \qquad A\boldsymbol{x} = \begin{pmatrix} \mathbf{0} \\ A_{i,1}x_1 \\ \boldsymbol{a}x_1 + A'\boldsymbol{x'} \end{pmatrix} \qquad A\boldsymbol{y} = \begin{pmatrix} \mathbf{0} \\ A_{i,1}y_1 \\ \boldsymbol{a}y_1 + A'\boldsymbol{y'} \end{pmatrix}$$

Here $A_{i,1}$ is the first positive entry at the i-th row with $1 \leqslant i < m$, $\boldsymbol{a}$ is a vector of length $m - i > 0$, and A' is an $(m-i) \times (n-1)$ echelon-form matrix. If $\boldsymbol{x} \geqslant^{\mathsf{lex}} \boldsymbol{y}$ is due to $x_1 > y_1$ then $A_{i,1}x_1 > A_{i,1}y_1$ and therefore $A\boldsymbol{x} \geqslant^{\mathsf{lex}} A\boldsymbol{y}$. Otherwise, $x_1 = y_1$ and $\boldsymbol{x'} \geqslant^{\mathsf{lex}} \boldsymbol{y'}$. By the induction hypothesis $A'\boldsymbol{x'} \geqslant^{\mathsf{lex}} A'\boldsymbol{y'}$ and therefore $A\boldsymbol{x} \geqslant^{\mathsf{lex}} A\boldsymbol{y}$. □

Lemma 19. *Let A be an $m \times n$ matrix. If $\boldsymbol{x} \geqslant^{\mathsf{lex}} \boldsymbol{y}$ implies $A\boldsymbol{x} \geqslant^{\mathsf{lex}} A\boldsymbol{y}$ for all vectors $\boldsymbol{x}, \boldsymbol{y}$ of natural numbers, then A is in echelon form.*

See Appendix A for the proof of Lemma 19. Let $\mathcal{M}$ be an algebra whose carrier is the set of vectors of natural numbers ordered lexicographically and interpretations $f_{\mathcal{M}}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = A_1\boldsymbol{x}_1 + \cdots + A_n\boldsymbol{x}_n + \boldsymbol{a}$ are built from echelon-form matrices $A_1, \ldots, A_n$. We dub such an algebra *echelon-form matrix interpretation*.

Theorem 20. *The pair $(\geqslant_{\mathcal{M}}, >_{\mathcal{M}})$ is a reduction pair for every echelon-form matrix interpretation $\mathcal{M}$.* □

To characterize strict monotonicity for $m \times n$ matrices A (i.e., the property that $\boldsymbol{x} >^{\mathsf{lex}} \boldsymbol{y}$ implies $A\boldsymbol{x} >^{\mathsf{lex}} A\boldsymbol{y}$), it suffices to additionally assume $n \leqslant m$ and $A_{i,i} > 0$ for all $1 \leqslant i \leqslant n$. Such an echelon-form matrix is called *positive*.

Example 21. The echelon-form matrix $A = \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right)$ is not positive, while A^T is. Indeed, A is not strictly monotone, as witnessed by $A \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) = A \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$.

Lemma 22. *Let A be an $m \times n$ positive echelon-form matrix. Then $A\boldsymbol{x} >^{\mathsf{lex}} A\boldsymbol{y}$ whenever $\boldsymbol{x} >^{\mathsf{lex}} \boldsymbol{y}$.*

Proof. We proceed by mathematical induction on n. The case $n = 1$ is trivial. Otherwise, write $\boldsymbol{x}, \boldsymbol{y}$ and A as follows:

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ \boldsymbol{x'} \end{pmatrix} \qquad \boldsymbol{y} = \begin{pmatrix} y_1 \\ \boldsymbol{y'} \end{pmatrix} \qquad A = \begin{pmatrix} A_{1,1} & \mathbf{0}^T \\ \boldsymbol{a} & A' \end{pmatrix}$$

Here $A_{1,1}$ is the positive entry, $\boldsymbol{a}$ is a vector of length $m-1$, and A' is a $(m-1) \times (n-1)$ positive echelon-form matrix. A calculation shows that

$$A\boldsymbol{x} = \begin{pmatrix} A_{i,1}x_1 \\ \boldsymbol{a}x_1 + A'\boldsymbol{x'} \end{pmatrix} \qquad A\boldsymbol{y} = \begin{pmatrix} A_{1,1}y_1 \\ \boldsymbol{a}y_1 + A'\boldsymbol{y'} \end{pmatrix}$$

If $\boldsymbol{x} >^{\mathsf{lex}} \boldsymbol{y}$ is by $x_1 > y_1$ then $A_{1,1}x_1 > A_{1,1}y_1$ and therefore $A\boldsymbol{x} >^{\mathsf{lex}} A\boldsymbol{y}$. Otherwise, $x_1 = y_1$ and $\boldsymbol{x'} >^{\mathsf{lex}} \boldsymbol{y'}$. By the induction hypothesis $A'\boldsymbol{x'} >^{\mathsf{lex}} A'\boldsymbol{y'}$ and therefore $A\boldsymbol{x} >^{\mathsf{lex}} A\boldsymbol{y}$. □

Lemma 23. *Let A be an $m \times n$ matrix. If $\boldsymbol{x} >^{\mathsf{lex}} \boldsymbol{y}$ implies $A\boldsymbol{x} >^{\mathsf{lex}} A\boldsymbol{y}$ for all vectors $\boldsymbol{x}, \boldsymbol{y}$ of natural numbers, then A is a positive echelon-form matrix.*

See Appendix A for the proof of Lemma 23. Now we can characterize monotone and invariant positions of echelon-form matrix interpretations $\mathcal{M}$. Let the interpretation of an n-ary function symbol f be $f_{\mathcal{M}}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = A_1\boldsymbol{x}_1 + \cdots + A_n\boldsymbol{x}_n + \boldsymbol{a}$. The i-th argument position of f is monotone if A_i is positive, and invariant if $A_i = O$. Besides, $(\geqslant_{\mathcal{M}}, >_{\mathcal{M}})$ is normal.

We again tame the Hydra, using an echelon-form matrix interpretation which cannot be simulated by combination of linear polynomial interpretations.

Example 24. Recall the TRS $\mathcal{H}$ in Section 4. Instead of the Knuth–Bendix order, we establish the termination by the combination of the ordinal interpretation $\mathcal{O}$ with the next echelon-form matrix interpretation $\mathcal{M}$:

$$\bullet_{\mathcal{M}}(\boldsymbol{x}) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \circ_{\mathcal{M}}(\boldsymbol{x}) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad [\!]_{\mathcal{M}}(\boldsymbol{x}) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

The other interpretations $\mathsf{0}_{\mathcal{M}}$, $\mathsf{H}_{\mathcal{M}}(\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}})$, $\mathsf{c}^1{}_{\mathcal{M}}(\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}})$, and $\mathsf{c}^2{}_{\mathcal{M}}(\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}}, \overline{\boldsymbol{z}})$ are defined as the constant vector $\mathbf{0}$. One can confirm the combinability, $\mathcal{R} \subseteq >_{\mathcal{OM}}$, and $\mathcal{E}\mathsf{mb} \subseteq \geqslant_{\mathcal{OM}}$. For instance, the orientations $[\!] \circ x >_{\mathcal{M}} \circ [\!]\ x$ and $\bullet [\!]\ x >_{\mathcal{M}} [\!] \bullet \bullet\ x$ of the first two rules in $\mathcal{H}$ are verified as follows:

$$1\colon \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} 8 \\ 0 \end{pmatrix} >^{\mathsf{lex}} \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} 7 \\ 0 \end{pmatrix} \quad 2\colon \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} >^{\mathsf{lex}} \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

So, the termination of $\mathcal{H}$ is again concluded.

The standard matrix interpretation with the component-wise order and our echelon-form matrix interpretation are incomparable. For example, the echelon-formed matrix interpretation can show the termination of Example 17 while the standard matrix interpretation cannot. For the converse, we consider the TRS

$\mathcal{R} = \{\mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{b}), \mathsf{g}(\mathsf{b}) \to \mathsf{g}(\mathsf{a})\}$. Then $\mathsf{DP}(\mathcal{R}) = \{\mathsf{f}^\sharp(\mathsf{a}) \to \mathsf{f}^\sharp(\mathsf{b}), \mathsf{g}^\sharp(\mathsf{b}) \to \mathsf{g}^\sharp(\mathsf{a})\}$. The next standard matrix interpretation $\mathcal{A}$ satisfies $\mathcal{R} \subseteq \geqslant_\mathcal{A}$ and $\mathsf{DP}(\mathcal{R}) \subseteq >_\mathcal{A}$:

$$\mathsf{a}_\mathcal{A} = \begin{pmatrix}1\\0\end{pmatrix} \qquad \mathsf{b}_\mathcal{A} = \begin{pmatrix}0\\1\end{pmatrix} \qquad \mathsf{f}_\mathcal{A}(\boldsymbol{x}) = \begin{pmatrix}1&1\\0&0\end{pmatrix}\boldsymbol{x} \qquad \mathsf{g}_\mathcal{A}(\boldsymbol{x}) = \begin{pmatrix}1&1\\1&1\end{pmatrix}\boldsymbol{x}$$

$$\mathsf{f}^\sharp_\mathcal{A}(\boldsymbol{x}) = \begin{pmatrix}1&1\\0&1\end{pmatrix}\boldsymbol{x} \qquad \mathsf{g}^\sharp_\mathcal{A}(\boldsymbol{x}) = \begin{pmatrix}0&1\\0&0\end{pmatrix}\boldsymbol{x}$$

Here, it is essential that a and b are incomparable with respect to the underlying component-wise order. Indeed, there is no echelon-form matrix interpretation $\mathcal{A}$ with $\mathcal{R} \subseteq \geqslant_\mathcal{A}$ and $\mathsf{DP}(\mathcal{R}) \subseteq >_\mathcal{A}$ due to totality of the lexicographic order: Any interpretation $\mathcal{A}$ satisfies $\mathsf{a} \geqslant_\mathcal{A} \mathsf{b}$ or $\mathsf{b} \geqslant_\mathcal{A} \mathsf{a}$. If $\mathsf{a} \geqslant_\mathcal{A} \mathsf{b}$ holds then $\mathsf{g}^\sharp(\mathsf{a}) \geqslant_\mathcal{A} \mathsf{g}^\sharp(\mathsf{b})$ by monotonicity. Similarly, if $\mathsf{b} \geqslant_\mathcal{A} \mathsf{a}$ then $\mathsf{f}^\sharp(\mathsf{b}) \geqslant_\mathcal{A} \mathsf{f}^\sharp(\mathsf{a})$. In either case it contradicts to $\mathsf{DP}(\mathcal{R}) \subseteq >_\mathcal{A}$.

6 Experiments

In order to evaluate the presented methods, we have implemented a prototype tool for proving termination and relative termination. The tool uses the dependency pair framework (Theorems 1 and 2) together with two standard refinements: an iterative cycle analysis based on strongly connected components in dependency graphs [1,10,13], and the usable rule criterion [14]. For relative termination, the tool uses the relative version of Theorem 1 (a generalization of Theorem 3), which cannot be used with the usable rule due to lack of minimality, see [15]. We compare the three classes of reduction pairs and their lexicographic combinations.

- L: the lexicographic path order [16] with argument filtering.
- E_d: echelon-form matrix interpretations on $\mathbb{N}^d$ with $0,1$-matrix coefficients equipped with the lexicographic order (Section 5).
- S_d: matrix interpretations on $\mathbb{N}^d$ with $0,1$-matrix coefficients equipped with the standard component-wise order [8]. Note that S_1 is the same as E_1.

Suitable precedences, argument filters, and interpretations are searched by the SMT solver Z3 [21]; see [4,27] for the SAT/SMT encoding techniques. The experiments were run on a PC with Intel Core i5-1340P CPU (4.6 GHz) and 8 GB memory with 60 seconds timeout for each (relative) termination problem.[1]

Table 1 summarizes the experimental results on 1528 termination problems in the TRS Standard category of the Termination Problem Database [24]. For instance, the numbers in column $\mathsf{E}_1\mathsf{L}$ are read as follows: In the aforementioned setting, lexicographic combinations of E_1 (linear interpretations) with L (the lexicographic path order with argument filtering) proved termination of 496 TRSs, while termination analysis on 13 TRSs did not finish within 60 seconds. In general, combination gives us more proofs. For example, while L and E_1 produce 512 proofs in total, the union of L, E_1, LE_1, and $\mathsf{E}_1\mathsf{L}$ amounts to 537 proofs.

[1] The tool and the full experimental data are available at https://www.jaist.ac.jp/project/saigawa/25cade/.

Table 1. Experiments on 1528 termination problems.

	L	LL	LLL	E_1	E_2	E_3	E_4	S_2	S_2S_2	$S_2S_2S_2$	E_1E_1	E_1L	LE_1
proved	372	389	387	477	562	566	564	596	619	617	506	496	406
timeout	*13*	*17*	*27*	*8*	*14*	*28*	*86*	*24*	*40*	*63*	*8*	*13*	*13*

Table 2. Experiments on 57 relative termination problems.

	L	LL	LLL	E_1	E_2	E_3	E_4	S_2	S_2S_2	$S_2S_2S_2$	E_1E_1	E_1L	LE_1
proved	4	20	22	8	43	45	47	10	47	47	41	30	27
timeout	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*

Table 2 summarizes the experimental results on relative termination. The TRS Relative category in the database contains 57 relative termination problems where Theorem 3 is available. Two problems of them are known to be relatively non-terminating. The union of E4 and LL amounts to 50 proofs. We note that the union subsumes 39 proofs by the powerful termination tool AProVE [12,17] which is a participant in the relative termination category of the Termination and Complexity Competition (termCOMP 2024). The following example is one of the remaining 11 problems.

Example 25. Consider the relative termination problem (`INVY_15/#3.42`) of $\mathcal{R}/\mathcal{S}$. Here $\mathcal{R}$ consists of the rules

$$\begin{aligned} \mathsf{half}(\mathsf{0}) &\to \mathsf{0} & \mathsf{lastbit}(\mathsf{0}) &\to \mathsf{0} \\ \mathsf{half}(\mathsf{s}(\mathsf{0})) &\to \mathsf{0} & \mathsf{lastbit}(\mathsf{s}(\mathsf{0})) &\to \mathsf{s}(\mathsf{0}) \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) & \mathsf{lastbit}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{lastbit}(x) \\ \mathsf{conv}(\mathsf{0}) &\to \mathsf{cons}(\mathsf{nil}, \mathsf{0}) & \mathsf{conv}(\mathsf{s}(x)) &\to \mathsf{cons}(\mathsf{conv}(\mathsf{half}(\mathsf{s}(x))), \mathsf{lastbit}(\mathsf{s}(x))) \end{aligned}$$

and $\mathcal{S} = \{\mathsf{rand}(x) \to x,\ \mathsf{rand}(x) \to \mathsf{rand}(\mathsf{s}(x))\}$. The set $\mathsf{DP}(\mathcal{R})$ consists of

$$\begin{aligned} \mathsf{half}^\sharp(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{half}^\sharp(x) & \mathsf{conv}^\sharp(\mathsf{s}(x)) &\to \mathsf{conv}^\sharp(\mathsf{half}(\mathsf{s}(x))) \\ \mathsf{lastbit}^\sharp(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{lastbit}^\sharp(x) & \mathsf{conv}^\sharp(\mathsf{s}(x)) &\to \mathsf{half}^\sharp(\mathsf{s}(x)) \\ & & \mathsf{conv}^\sharp(\mathsf{s}(x)) &\to \mathsf{lastbit}^\sharp(\mathsf{s}(x)) \end{aligned}$$

The following echelon-form matrix interpretation $\mathcal{A}$ on $\mathbb{N}^3$ satisfies $\mathsf{DP}(\mathcal{R}) \subseteq >_{\mathcal{A}}$ and $\mathcal{R} \cup \mathcal{S} \subseteq \geqslant_{\mathcal{A}}$.

$$\begin{aligned} &\mathsf{0}_{\mathcal{A}} = \mathbf{0} & &\mathsf{nil}_{\mathcal{A}} = \mathbf{0} & &\mathsf{lastbit}_{\mathcal{A}}(\boldsymbol{x}) = \boldsymbol{e}_1 & &\mathsf{conv}_{\mathcal{A}}(\boldsymbol{x}) = \mathbf{0} \\ &\mathsf{s}_{\mathcal{A}}(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{e}_2 & &\mathsf{cons}_{\mathcal{A}}(\boldsymbol{x}, \boldsymbol{y}) = \mathbf{0} & &\mathsf{lastbit}^\sharp_{\mathcal{A}}(\boldsymbol{x}) = \boldsymbol{x} & &\mathsf{conv}^\sharp_{\mathcal{A}}(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{e}_1 \\ &\mathsf{half}_{\mathcal{A}}(\boldsymbol{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{x} & &\mathsf{half}^\sharp_{\mathcal{A}}(\boldsymbol{x}) = \boldsymbol{x} & &\mathsf{rand}_{\mathcal{A}}(\boldsymbol{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{x} + \boldsymbol{e}_1 \end{aligned}$$

Since $\mathcal{R}$ dominates $\mathcal{S}$ and $\mathcal{S}$ is non-duplicating, the above inclusions together with Theorem 3 entail the termination of $\mathcal{R}/\mathcal{S}$.

7 Conclusion

We have presented a simple criterion for combining reduction pairs based on monotone and invariant positions. Moreover, we have elucidated when the matrix interpretation with the lexicographic order induces a reduction pair. We conclude the paper by stating related work and future work.

Our combinability criterion (Theorem 10) is inspired by Touzet's work [25]. It is easy to confirm the precise correspondence between her interpretation $\mathcal{A}$ for the termination of the Hydra and Example 24. For instance, $\bullet_{\mathcal{A}}((x, m, n)) = (x, m, m + n + 1)$ corresponds to $\bullet_{\mathcal{O}}(x) = x$ and $\bullet_{\mathcal{M}}(\boldsymbol{x}) = \left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)\boldsymbol{x} + \left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right)$. We anticipate that the use of lexicographic combination eases termination analysis of challenging rewrite systems such as Goodstein sequences [28].

The ordinal interpretation in a certain form is the echelon-form matrix interpretation and lexicographic combination of the linear polynomial interpretation. To see this, consider linear polynomial interpretations

$$f_{\mathcal{A}}(x_1, \ldots, x_n) = a_0 + \sum_i a_i x_i \qquad f_{\mathcal{B}}(y_1, \ldots, y_n) = b_0 + \sum_i b_i y_i$$

with $a_i, b_i \in \mathbb{N}$ and the interpretation $f_{\mathcal{O}}$ on ordinal numbers below ω^2 given by:

$$f_{\mathcal{O}}(\omega x_1 + y_1, \ldots, \omega x_n + y_1) = \omega(a_0 + a_1 x_1 + \cdots + a_n x_n) + (b_0 + b_1 y_1 + \cdots + b_n y_n)$$

Here x_i and y_i range over $\mathbb{N}$. Then, $(\geqslant_{\mathcal{O}}, >_{\mathcal{O}})$ is order isomorphic to $(\geqslant_{\mathcal{AB}}, >_{\mathcal{AB}})$. In general, an n-times combination of the linear polynomial interpretation corresponds to an ordinal interpretation below ω^n. For instance, the echelon-form matrix interpretation $\mathsf{rand}_{\mathcal{A}}$ of Example 25 corresponds to $\mathsf{rand}_{\mathcal{O}}(\omega^2 x_1 + \omega x_2 + x_3) = \omega^2(x_1 + 1)$ with $x_1, x_2, x_3 \in \mathbb{N}$. Actually, it is equivalent to $\mathsf{rand}_{\mathcal{O}}(x) = x + \omega^2$ with $x \in \omega^3$.

There is another way to combine reduction pairs lexicographically. Given two reduction pairs $(\geqslant_1, >_1)$ and $(\geqslant_2, >_2)$, the pair $(\geqslant_1 \cap \geqslant_2, >_{12})$ forms a reduction pair, where $>_{12}$ is defined as in Definition 5. In contrast to Theorem 10, this construction imposes neither monotonicity nor invariance on the underlying reduction pairs. However, Theorem 2 with $(\geqslant_1 \cap \geqslant_2, >_{12})$ is simulated by successive application of Theorem 2 with $(\geqslant_1, >_1)$ and $(\geqslant_2, >_2)$.

The *rule removal* method [12] employs a monotone reduction pair to eliminate rules from $\mathcal{R}$ of a dependency pair problem $(\mathcal{P}, \mathcal{R})$. Considering the fact that combinability is a generalization of monotonicity, we anticipate that investigation on relationship of combinability and finiteness leads to a generalization of the method.

Adapting the echelon-form matrix interpretation for AC termination is another direction for future work. As shown in [3,20], lexicographic combination of *non-linear* polynomial interpretations is an effective proof method for AC termination. We believe that the non-linear matrix interpretation [5], using matrices instead of vectors, is a key for the work.

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References

1. Arts, T., Giesl, J.: Termination of term rewriting using dependency pairs. Theoretical Computer Science **236**, 133–178 (2000). https://doi.org/10.1016/S0304-3975(99)00207-8
2. Baader, F., Nipkow, T.: Term Rewriting and All That. Cambridge University Press (1998). https://doi.org/10.1017/CBO9781139172752
3. Ben Cherifa, A., Lescanne, P.: Termination of rewriting systems by polynomial interpretations and its implementation. Science of Computer Programming **9**(2), 137–159 (1987). https://doi.org/10.1016/0167-6423(87)90030-X
4. Codish, M., Giesl, J., Schneider-Kamp, P., Thiemann, R.: SAT solving for termination proofs with recursive path orders and dependency pairs. Journal of Automated Reasoning **49**, 53–93 (2012). https://doi.org/10.1145/1890028.1890030
5. Courtieu, P., Gbedo, G., Pons, O.: Improved matrix interpretation. In: Proc. 36th International Conference on Theory and Practice of Computer Science. LNCS, vol. 5901, pp. 283–295 (2010). https://doi.org/10.1007/978-3-642-11266-9_24
6. Dershowitz, N.: Orderings for term-rewriting systems. Theoretical Computer Science **17**, 279–301 (1982). https://doi.org/10.1016/0304-3975(82)90026-3
7. Dershowitz, N., Moser, G.: The Hydra battle revisited. In: Rewriting, Computation and Proof, Essays Dedicated to Jean-Pierre Jouannaud on the Occasion of his 60th Birthday. LNCS, vol. 4600, pp. 1–27 (2007). https://doi.org/10.1007/978-3-540-73147-4_1
8. Endrullis, J., Waldmann, J., Zantema, H.: Matrix interpretations for proving termination of term rewriting. Journal of Automated Reasoning **40**(2-3), 195–220 (2008). https://doi.org/10.1007/S10817-007-9087-9
9. Geser, A.: Relative Termination. Ph.D. thesis, University of Passau, Germany (1990), http://vts.uni-ulm.de/docs/2012/8146/vts_8146_11884.pdf
10. Giesl, J., Arts, T., Ohlebusch, E.: Modular termination proofs for rewriting using dependency pairs. Journal of Symbolic Computation **34**, 21–58 (2002). https://doi.org/doi.org/10.1006/jsco.2002.0541
11. Giesl, J., Thiemann, R., Schneider-Kamp, P.: The dependency pair framework: Combining techniques for automated termination proofs. In: Proc. 11th International Conference on Logic Programming and Automated Reasoning. LNCS (LNAI), vol. 3452, pp. 301–331 (2005). https://doi.org/10.1007/978-3-540-32275-7_21
12. Giesl, J., Thiemann, R., Schneider-Kamp, P., Falke, S.: Automated termination proofs with AProVE. In: Proc. 15th International Conference on Rewriting Techniques and Applications. LNCS, vol. 3091, pp. 210–220 (2004). https://doi.org/10.1007/978-3-540-25979-4_15
13. Hirokawa, N., Middeldorp, A.: Automating the dependency pair method. Information and Computation **199**(1-2), 172–199 (2005). https://doi.org/10.1016/J.IC.2004.10.004
14. Hirokawa, N., Middeldorp, A.: Tyrolean termination tool: Techniques and features. Information and Computation **205**(4), 474–511 (2007). https://doi.org/10.1016/J.IC.2006.08.010
15. Iborra, J., Nishida, N., Vidal, G., Yamada, A.: Relative termination via dependency pairs. Journal of Automated Reasoning **58**, 391–411 (2017). https://doi.org/10.1007/s10817-016-9373-5
16. Kamin, S., Lévy, J.: Two generalizations of the recursive path ordering. Tech. rep., University of Illinois (1980), unpublished manuscript.

17. Kassing, J.C., Vartanyan, G., Giesl, J.: A dependency pair framework for relative termination of term rewriting. In: Proc. 12th International Joint Conference on Automated Reasoning. LNCS, vol. 14740, pp. 360–380 (2024). https://doi.org/10.1007/978-3-031-63501-4_19
18. Kirby, L., Paris, J.: Accessible independence results for Peano arithmetic. Bulletin of the London Mathematical Society **14**, 285–325 (1982). https://doi.org/10.1112/blms/14.4.285
19. Knuth, D., Bendix, P.: Simple word problems in universal algebras. In: Leech, J. (ed.) Computational Problems in Abstract Algebra, pp. 263–297. Pergamon Press (1970)
20. Lankford, D.: On proving term rewriting systems are noetherian. Tech. rep., Louisiana Technical University (1979)
21. de Moura, L.M., Bjørner, N.S.: Z3: an efficient SMT solver. In: Proc. 14th International Conference on Tools and Algorithms for the Construction and Analysis of Systems. LNCS, vol. 4963, pp. 337–340 (2008). https://doi.org/10.1007/978-3-540-78800-3_24
22. Neurauter, F., Middeldorp, A.: Revisiting matrix interpretations for proving termination of term rewriting. In: Proc. 22nd International Conference on Rewriting Techniques and Applications. LIPIcs, vol. 10, pp. 251–266 (2011). https://doi.org/10.4230/LIPICS.RTA.2011.251
23. Terese: Term Rewriting Systems. Cambridge University Press (2003)
24. Termination Community: The Termination Problem Database (TPDB). https://github.com/TermCOMP/TPDB, accessed: 05.12.2023
25. Touzet, H.: Encoding the Hydra battle as a rewrite system. In: Proc. 23rd International Symposium on Mathematical Foundations of Computer Science. LNCS, vol. 1450, pp. 267–276 (1998). https://doi.org/10.1007/BFb0055776
26. Yamada, A., Kusakari, K., Sakabe, T.: A unified ordering for termination proving. Science of Computer Programming **111**, 110–134 (2015). https://doi.org/10.1016/j.scico.2014.07.009
27. Zankl, H., Hirokawa, N., Middeldorp, A.: KBO orientability. Journal of Automated Reasoning **43**, 173–201 (2009). https://doi.org/10.1007/s10817-009-9131-z
28. Zankl, H., Winkler, S., Middeldorp, A.: Beyond polynomials and peano arithmetic - automation of elementary and ordinal interpretations. Journal of Symbolic Computation **69**, 129–158 (2015). https://doi.org/10.1016/J.JSC.2014.09.033

A Omitted Examples and Proofs

The next example shows that the normality requirement cannot be dropped from the combinability criterion (Theorem 10)

Example 26. Consider the linear polynomial interpretation $\mathcal{A}$ defined by with $\mathsf{a}_\mathcal{A} = 1$, $\mathsf{b}_\mathcal{A} = 0$, and $\mathsf{f}_\mathcal{A}(x) = 0$. With using the identity relation $=$, the reduction pairs $(=, >_\mathcal{A})$ and $(\geqslant_\mathcal{A}, >_\mathcal{A})$ satisfy all conditions for combinability except normality, as witnessed by $\mathsf{a} >_\mathcal{A} \mathsf{b}$ and $\mathsf{a} \neq \mathsf{b}$. Indeed, the preorder $\geqslant$ of the lexicographic combination is not closed under contexts, because $\mathsf{a} \geqslant \mathsf{b}$ from $\mathsf{a} >_\mathcal{A} \mathsf{b}$ but not $\mathsf{f}(\mathsf{a}) \geqslant \mathsf{f}(\mathsf{b})$, as neither $\mathsf{f}(\mathsf{a}) >_\mathcal{A} \mathsf{f}(\mathsf{b})$ nor $\mathsf{f}(\mathsf{a}) = \mathsf{f}(\mathsf{b})$ holds.

The next example shows that the non-emptiness requirement cannot be dropped from Theorem 14.

Example 27. Consider the following linear polynomial interpretations $\mathcal{A}$ and $\mathcal{B}$:

$$\mathsf{f}_{\mathcal{A}}(x) = 0 \qquad \mathsf{a}_{\mathcal{A}} = 1 \qquad \mathsf{b}_{\mathcal{A}} = 0 \qquad\qquad \mathsf{f}_{\mathcal{B}}(x) = 0 \qquad \mathsf{a}_{\mathcal{B}} = 0 \qquad \mathsf{b}_{\mathcal{B}} = 0$$

When the signature consists of the only three symbols f, a, and b, the relation $>_{\mathcal{B}}$ is the empty relation. Therefore, the first argument position of f is monotone with respect to $>_{\mathcal{B}}$. However, the position is not monotone with respect to the strict order $>_{\mathcal{AB}}$ of the lexicographic combination, as $>_{\mathcal{AB}}$ degenerates to $>_{\mathcal{A}}$.

Proof (of Lemma 19). We show that every element $A_{i,j}$ with $2 \leqslant j \leqslant n$ satisfies the following property: If $A_{k,\,j-1} = 0$ for all $k < i$ then $A_{i,j} = 0$. We proceed by complete induction on i while fixing j with $2 \leqslant j \leqslant n$. Suppose $A_{i',j-1} = 0$ for all $i' < i$. By the induction hypothesis $A_{i',j} = 0$ for all $i' < i$, too. Assume to the contrary $A_{i,j} > 0$. For the vectors $\boldsymbol{x} = A_{i,j}\boldsymbol{e}_{j-1}$ and $\boldsymbol{y} = (A_{i,j-1} + 1)\boldsymbol{e}_j$ the inequality $\boldsymbol{x} \geqslant^{\mathsf{lex}} \boldsymbol{y}$ holds. One can deduce the implications:

$$\boldsymbol{x} \geqslant^{\mathsf{lex}} \boldsymbol{y} \implies A\boldsymbol{x} \geqslant^{\mathsf{lex}} A\boldsymbol{y} \implies A_{i,j-1}A_{i,j} \geqslant A_{i,j-1}A_{i,j} + A_{i,j} \implies 0 \geqslant A_{i,j}$$

Thus, the desired contradiction $0 \geqslant A_{i,j} > 0$ is obtained. □

Proof (of Lemma 23). From Lemma 19, we know that A is in echelon form. If $n > m$, then the right-most column of A is all zero, which implies $A\boldsymbol{e}_n = \mathbf{0} = A\mathbf{0}$, a contradiction. So $n \leqslant m$. Notice that from the echelon formedness, $A_{i,i} > 0$ implies $A_{(i-1),(i-1)} > 0$ for all $2 \leqslant i \leqslant n$. So, it suffices to see $A_{n,n} > 0$, which is again shown by contradiction: If $A_{n,n} = 0$, then $A\boldsymbol{e}_n = \mathbf{0} = A\mathbf{0}$. □

We also note that an echelon-form square matrix A is positive if and only if it satisfies *weak simplicity* $A\boldsymbol{x} \geqslant^{\mathsf{lex}} \boldsymbol{x}$, which is a relevant property for the weighted path order [26].