Non-deterministic inductive definitions and Fullness

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Abstract

In this paper, we deal with the non-deterministic inductive definition principle NID with the weak notion of a set-generated class introduced by van den Berg and with the strong notion of a set-generated class adopted by Aczel et al.. We introduce a principle, called nullary NID, and prove that nullary NID is equivalent to Fullness in a subsystem of the constructive Zermelo-Fraenkel set theory.

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1 Introduction

The notion of a set-generated class was introduced in Aczel [4] for dcpos using some terminology from domain theory. A partially ordered class is a directed complete partial order (dcpo) if each directed subset has a least upper bound, where a subset is directed if any pair of elements of the subset has an upper bound in the subset. A dcpo $X$ is set-generated if there is a subset $G$ of $X$ such that, for each $a \in X$, $\{x \in G \mid x \leq a\}$ is a directed subset whose least upper bound is $a$. If we restrict our attention to a class $X$ of subsets of a set with the inclusion $\subseteq$ as a partial order, then we may say that $X$ is set-generated if there exists a subset $G$ of $X$ such that

$$\forall \alpha \in X \forall \tau \in \text{Fin}(\alpha) \exists \beta \in G[\tau \subseteq \beta \subseteq \alpha],$$
where Fin(α) is the set of finitely enumerable subsets of α.

This definition was adopted in Aczel et al. [5], and it was shown that the notion of a set-generated class plays crucial roles in predicative constructive mathematics. However, in an early draft of [5] and in van den Berg [8], a weaker notion of a set-generated class was employed: a class X of subsets of a set is set-generated if there exists a subset G of X such that

\[ \forall \alpha \in X \forall x \in \alpha \exists \beta \in G [x \in \beta \subseteq \alpha]. \]

Note that the strong notion of a set-generated class is crucial in some applications, for example, [9].

In [8], van den Berg introduced the principle NID on non-deterministic inductive definitions and set-generated classes using the weaker notion of a set-generated class in the constructive set theory CZF. A rule on a set S is a pair (a, b) of subsets a and b of S, and a rule is called elementary if a is a singleton and finitary if a is finitely enumerable. A subset α of S is closed under the rule (a, b) if

\[ a \subseteq \alpha \Rightarrow b \not\subseteq \alpha, \]

where \( b \not\subseteq \alpha \Leftrightarrow \exists x \in b (x \in \alpha) \), that is, the intersection of b and α is inhabited. For a set R of rules on S, we call a subset α of S R-closed if it is closed under each rule in R. The NID principle is that for each set S and set R of rules on S, the class of R-closed subsets of S is set-generated. If we restrict rules in NID to elementary and finitary rules, we call them elementary NID and finitary NID, respectively.

On the other hand, in [5], Aczel et al. characterized set-generated classes with the strong notion using generalized geometric theories and the set generation axiom (SGA) in CZF. In [8], van den Berg discussed on a relation between finitary NID and SGA, and revealed some aspect between the weak notion and the strong notion of a set-generated class. He also showed that elementary NID implies Fullness, a theorem in CZF, which is an important axiom in a subsystem of CZF that implies Exponentiation (the class of functions between sets is a set).

In this paper, we introduce another NID principle, called nullary NID, which is weaker than finitary NID, and prove that nullary NID is equivalent to Fullness in a subsystem of CZF, that is, the elementary constructive set theory ECST. We also show that elementary NID implies nullary NID, and that nullary, elementary and finitary NID with the weak notion of a set-generated class are equivalent to respectively nullary, elementary and finitary NID with the strong notion.
2 The elementary constructive set theory

The constructive Zermelo-Fraenkel set theory CZF, founded by Aczel [1, 2, 3], grew out of Myhill’s constructive set theory [11] as a formal system for Bishop’s constructive mathematics, and permits a quite natural interpretation in Martin-Löf type theory [10]. Aczel and Rathjen introduced the elementary constructive set theory ECST which is a subsystem of CZF in their book draft [7] written by extending their research report [6].

Definition 1. The language of a constructive set theory contains variables for sets and the binary predicates = and ∈. The axioms and rules are those of intuitionistic predicate logic with equality. In addition, ECST has the following set theoretic axioms:

Extensionality: ∀a∀b(∀x(x ∈ a ↔ x ∈ b) → a = b).

Pairing: ∀a∀b∃c∀x(x ∈ c ↔ x = a ∨ x = b).

Union: ∀a∃b∀x(x ∈ b ↔ ∃y(x ∈ a)).

Restricted Separation:
∀a∃b∀x(x ∈ b ↔ x ∈ a ∧ ϕ(x))

for every restricted formula ϕ(x). Here a formula ϕ(x) is restricted, or Δ₀, if all the quantifiers occurring in it are bounded, i.e. of the form ∀x ∈ c or ∃x ∈ c.

Replacement:
∀a(∀x ∈ a∃!yϕ(x, y) → ∃b∀y(y ∈ b ↔ ∃x ∈ aϕ(x, y)))

for every formula ϕ(x, y).

Strong Infinity:
∃a[0 ∈ a ∧ ∀x(x ∈ a → x + 1 ∈ a) ∧ ∀y(0 ∈ y ∧ ∀x(x ∈ y → x + 1 ∈ y) → a ⊆ y)],

where x + 1 is x ∪ {x}, and 0 is the empty set ∅.
Let $a$ and $b$ be sets. Using Replacement and Union, the cartesian product $a \times b$ of $a$ and $b$ consisting of the ordered pairs $(x, y) = \{ \{x\}, \{x, y\}\}$ with $x \in a$ and $y \in b$ can be introduced in ECST. Similarly, we can introduce the disjoint union $a + b = \{(0, x) \mid x \in a\} \cup \{(1, y) \mid y \in b\}$, where $1 = \{0\}$. A relation $r$ between $a$ and $b$ is a subset of $a \times b$. A relation $r \subseteq a \times b$ is total (or is a multivalued function) if for every $x \in a$ there exists $y \in b$ such that $(x, y) \in r$. The class of total relations between $a$ and $b$ is denoted by $\text{mv}(a, b)$, or more formally

$$r \in \text{mv}(a, b) \iff r \subseteq a \times b \land \forall x \in a \exists y \in b((x, y) \in r).$$

A function from $a$ to $b$ is a total relation $f \subseteq a \times b$ such that for every $x \in a$ there is exactly one $y \in b$ with $(x, y) \in f$. The class of functions from $a$ to $b$ is denoted by $b^a$, or more formally

$$f \in b^a \iff f \in \text{mv}(a, b) \land \forall x \in a \forall y, z \in b((x, y) \in f \land (x, z) \in f \to y = z).$$

We use $\omega$ for the unique set $a$ such that $0 \in a \land \forall x (x \in a \to x + 1 \in a)$, ensured by Strong Infinity. For a set $S$, let $\text{Fin}(S)$ denote the class $\{\text{ran}(f) \mid f \in S^n, n \in \omega\}$ of finitely enumerable subsets of $S$, and let $\text{Fin}^+(S)$ denote the class $\{\sigma \in \text{Fin}(S) \mid \exists x \in S(x \in \sigma)\}$ of finitely enumerable inhabited subsets of $S$.

The constructive set theory CZF is obtained from ECST by replacing Replacement by

**Strong Collection:**

$$\forall a (\forall x \in a \exists y \varphi(x, y) \to \exists b (\forall x \in a \exists y \in b \varphi(x, y) \land \forall y \in b \exists x \in a \varphi(x, y)))$$

for every formula $\varphi(x, y)$,

and adding

**Subset Collection:**

$$\forall a \forall b \exists c \forall u (\forall x \in a \exists y \in b \varphi(x, y, u) \to$$

$$\exists d \in c (\forall x \in a \exists y \in d \varphi(x, y, u) \land \forall y \in d \exists x \in a \varphi(x, y, u)))$$

for every formula $\varphi(x, y, u)$, and

**$\in$-Induction:** $\forall a (\forall x \in a \varphi(x) \to \varphi(a)) \to \forall a \varphi(a)$, for every formula $\varphi(a)$.  

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In ECST, Subset Collection implies

**Fullness:** \( \forall a \forall b \exists c(c \subseteq \text{mv}(a, b) \land \forall r \in \text{mv}(a, b) \exists s \in c(s \subseteq r)) \),

and Fullness and Strong Collection imply Subset Collection. The notable consequence of Fullness is that \( b^a \) forms a set, that is

**Exponentiation:** \( \forall a \forall b \exists c \forall f(f \in c \leftrightarrow f \in b^a) \).

Note that, in the presence of Exponentiation (and hence Fullness), the classes \( \text{Fin}(S) \) and \( \text{Fin}^+(S) \) are sets for each set \( S \), by Replacement, Union and Restricted Separation.

For a set \( S \), we write \( \text{Pow}(S) \) for the power class of \( S \) which is not a set in ECST nor in CZF:

\[ a \in \text{Pow}(S) \leftrightarrow a \subseteq S. \]

### 3 Non-deterministic inductive definitions and Fullness

In this section, we work within the subsystem ECST of CZF.

**Definition 2.** Let \( S \) be a set. Then a **rule** on \( S \) is a pair \( (a, b) \) of subsets \( a \) and \( b \) of \( S \). A rule is called **nullary** if \( a \) is empty, **elementary** if \( a \) is a singleton and **finitary** if \( a \) is finitely enumerable. A subset \( \alpha \) of \( S \) is **closed under** the rule \( (a, b) \) if

\[ a \subseteq \alpha \Rightarrow b \not\subseteq \alpha. \]

For a set \( R \) of rules on \( S \), we call a subset \( \alpha \) of \( S \) **\( R \)-closed** if it is closed under each rule in \( R \).

**Remark 3.** Note that if a rule is nullary or elementary, then it is finitary.

**Definition 4.** Let \( S \) be a set and let \( X \) be a subclass of \( \text{Pow}(S) \). Then \( X \) is **set-generated** if there exists a subset \( G \) of \( X \) such that

\[ \forall \alpha \in X \forall x \in \alpha \exists \beta \in G(x \in \beta \subseteq \alpha), \]

and **strongly set-generated** if there exists a subset \( G \) of \( X \) such that

\[ \forall \alpha \in X \forall \sigma \in \text{Fin}(\alpha) \exists \beta \in G(\sigma \subseteq \beta \subseteq \alpha). \]
Definition 5. Let NID and NID* denote the principles that for each set S and set R of rules on S, the class of R-closed subsets of S is set-generated and strongly set-generated, respectively. The principles obtained by restricting R in NID to a set of nullary, elementary and finitary rules are called nullary NID, elementary NID* and finitary NID*, respectively.

Remark 6. Trivially, NID* implies NID, and note that finitary NID* implies nullary NID and elementary NID*.

For a set S, let *S denote the set \{x ∈ S | x /∈ S\} which is not in S.

Theorem 7. Nullary NID is equivalent to Fullness.

Proof. Suppose nullary NID. Let A and B be sets, and define a set R of nullary rules on (A × B) ∪ \{*AxB\} by

\[ R = \{(\emptyset, \{x\} × B) | x ∈ A\} ∪ \{(\emptyset, \{*AxB\})\}. \]

Then, by nullary NID, there exists a subset G of the class X of R-closed subsets of (A × B) ∪ \{*AxB\} such that

\[ ∀α ∈ X∀z ∈ α∃β ∈ G(z ∈ β ⊆ α). \]

Let C = \{β ∩ (A × B) | β ∈ G\}. Then for each β ∈ G and x ∈ A, since β /∈ \{x\} × B, there exists y ∈ B such that (x, y) ∈ β, and hence (x, y) ∈ β ∩ (A × B). Therefore C ⊆ mv(A, B). For each r ∈ mv(A, B), since \{x\} × B /∈ r ∪ \{*AxB\} for each x ∈ A and *AxB ∈ r ∪ \{*AxB\}, we have r ∪ \{*AxB\} ∈ X, and hence there exists β ∈ G such that *AxB ∈ β ⊆ r ∪ \{*AxB\}. Thus β ∩ (A × B) ⊆ (r ∪ \{*AxB\}) ∩ (A × B) = r.

Conversely, suppose Fullness. Let R be a set of nullary rules on a set S. Then, by Fullness, there exists a set C ⊆ mv(R, S) such that

\[ ∀r ∈ mv(R, S)∃s ∈ C(s ⊆ r). \]

Let

\[ G = \{\{x\} ∪ \text{ran}(r) | x ∈ S, r ∈ C, ∀((\emptyset, b), y) ∈ r(y ∈ b)\}. \]

If r ∈ C and ∀((\emptyset, b), y) /∈ r(y ∈ b), then for each (\emptyset, b) ∈ R, we have b /∈ \text{ran}(r), and hence b /∈ \{x\} ∪ \text{ran}(r) for each x ∈ S. Therefore G is a subset of the class of R-closed subsets of S. For each R-closed subset α of S, since r = \{((\emptyset, b), y) | (\emptyset, b) ∈ R, y ∈ b ∩ α\} ∈ mv(R, S), there exists s ∈ C such that s ⊆ r. Note that, since ∀((\emptyset, b), y) ∈ r(y ∈ b) and ran(r) ⊆ α, we have ∀((\emptyset, b), y) ∈ s(y ∈ b) and ran(s) ⊆ α. Therefore if x ∈ α, then \{x\} ∪ ran(s) ∈ G and x ∈ \{x\} ∪ ran(s) ⊆ \{x\} ∪ α = α. □
Proposition 8. Elementary NID implies nullary NID.

Proof. Let $R$ be a set of nullary rules on a set $S$, and define a set $R'$ of elementary rules on $S \cup \{S\}$ by

$$R' = \{\{(S), b\} \mid (\emptyset, b) \in R\} \cup \{(\{x\}, \{S\})\} \mid x \in S\}.$$ 

Then, by elementary NID, there exists a subset $G'$ of the class $X'$ of $R'$-closed subsets of $S \cup \{S\}$ such that

$$\forall \alpha' \in X' \forall z \in \alpha' \exists \beta' \in G'(z \in \beta' \subseteq \alpha').$$

Let $G = \{\beta' \cap S \mid \beta' \in G', S \in \beta'\}$. Then for each $\beta' \cap S \in G$ and $(\emptyset, b) \in R$, since $S \in \beta'$, we have $b \not\in \beta'$, and hence $b \not\in \beta' \cap S$. Therefore $G$ is a set of $R$-closed subsets of $S$. Let $\alpha$ be an $R$-closed subset of $S$. Then for each $(\emptyset, b) \in R$, since $b \not\in \alpha$, we have $b \not\in \alpha \cup \{S\}$, and $S \in \alpha \cup \{S\}$. Hence $\alpha \cup \{S\} \in X'$. Therefore for each $x \in \alpha$, since $x \in \alpha \cup \{S\}$, there exists $\beta' \in G'$ such that $x \in \beta' \subseteq \alpha \cup \{S\}$, and so $S \in \beta'$. Thus $\beta' \cap S \in G$ and $x \in \beta' \cap S \subseteq (\alpha \cup \{S\}) \cap S = \alpha$. \hfill \Box

Remark 9. Note that, assuming nullary, elementary or finitary NID, we have Fullness, and hence the classes $\text{Fin}(S)$ and $\text{Fin}^+(S)$ are sets for each set $S$.

For a subset $\alpha$ of the disjoint union $A + B$, let $(\alpha)_0$ and $(\alpha)_1$ denote the sets $\{x \in A \mid (0, x) \in \alpha\}$ and $\{y \in B \mid (1, y) \in \alpha\}$, respectively.

Proposition 10. Nullary, elementary and finitary NID imply nullary, elementary and finitary NID*, respectively.

Proof. Let $R$ be a set of nullary rules on a set $S$, and define a set $R'$ of nullary rules on $\text{Fin}(S)$ by

$$R' = \{\emptyset, \text{Fin}^+(b)\} \mid (\emptyset, b) \in R\}.$$ 

Then, by nullary NID, there exists a subset $G'$ of the class $X'$ of $R'$-closed subsets of $\text{Fin}(S)$ such that

$$\forall \alpha' \in X' \forall \sigma \in \alpha' \exists \beta' \in G'(\sigma \in \beta' \subseteq \alpha').$$

Let $G = \{\bigcup \beta' \mid \beta' \in G'\}$. Then for each $\beta' \in G'$ and $(\emptyset, b) \in R$, since $\text{Fin}^+(b) \not\in \beta'$, there exists $\sigma \in \text{Fin}^+(b) \cap \beta'$, hence there exists $x \in b$ such that $x \in \sigma \in \beta'$, and therefore $b \not\in \bigcup \beta'$. Thus $G$ is a set of $R$-closed subsets
of $S$. Let $\alpha$ be an $R$-closed subset of $S$. Then for each $(\emptyset, \text{Fin}^+ (b)) \in R'$, since $(\emptyset, b) \in R$, we have $b \not\in \alpha$, and hence $\text{Fin}^+ (b) \not\in \text{Fin} (\alpha)$. Therefore $\text{Fin} (\alpha) \in X'$. Thus for each $\sigma \in \text{Fin} (\alpha)$ there exists $\beta' \in G'$ such that $\sigma \in \beta' \subseteq \text{Fin} (\alpha)$, and so $\sigma \subseteq \bigcup \beta' \subseteq \bigcup \text{Fin} (\alpha) = \alpha$.

Let $R$ be a set of elementary or finitary rules on a set $S$, and define a set $R'$ of elementary and finitary rules on $S + \text{Fin} (S)$ by

$$R' = \{(1 \times a, 1 \times b) \mid (a, b) \in R\} \cup \{(1 \times a, \{(1, \sigma) \mid \sigma \in \text{Fin} (b)\}) \mid (a, b) \in R\} \cup \{\{(1, \sigma), \{(0, x)\}) \mid \sigma \in \text{Fin} (S), x \in \sigma\}.$$ 

Note that $R'$ is elementary and finitary if $R$ is elementary and finitary, respectively. Then, by elementary or finitary NID, there exists a subset $G'$ of the class $X'$ of $R'$-closed subsets of $S + \text{Fin} (S)$ such that

$$\forall \alpha' \in X' \forall z \in \alpha' \exists \beta' \in G' (z \in \beta' \subseteq \alpha').$$

Let $G = \{(\beta')_0 \mid \beta' \in G'\}$. Then for each $\beta' \in G'$ and $(a, b) \in R$ with $a \subseteq (\beta')_0$, since $1 \times a \subseteq \beta'$, we have $1 \times b \not\in \beta'$, and hence $b \not\in (\beta')_0$. Therefore $G$ is a set of $R$-closed subsets of $S$. Let $\alpha$ be an $R$-closed subset of $S$. Then $\alpha + \text{Fin} (\alpha)$ is an $R'$-closed subset of $S + \text{Fin} (S)$. In fact, for each $(a, b) \in R$, if $1 \times a \subseteq \alpha + \text{Fin} (\alpha)$, then, since $a \subseteq \alpha$, we have $b \not\in \alpha$, and hence $1 \times b \not\in \alpha + \text{Fin} (\alpha)$ and $(1, \sigma) \in \alpha + \text{Fin} (\alpha)$ for some $\sigma \in \text{Fin} (b)$. For $\{(1, \sigma), \{(0, x)\}) \in R'$ where $\sigma \in \text{Fin} (S)$ and $x \in \sigma$, if $\{(1, \sigma)\} \subseteq \alpha + \text{Fin} (\alpha)$, then, since $\sigma \in \text{Fin} (\alpha)$, we have $x \in \sigma \subseteq \alpha$, and hence $(0, x) \in \alpha + \text{Fin} (\alpha)$. Therefore $\alpha + \text{Fin} (\alpha) \in X'$. Thus for each $\sigma \in \text{Fin} (\alpha)$ there exists $\beta' \in G'$ such that $(1, \sigma) \in \beta' \subseteq \alpha + \text{Fin} (\alpha)$, and since $\beta'$ is $R'$-closed and $(0, x) \in \beta'$ for each $x \in \sigma$, we have $\sigma \subseteq (\beta')_0 \subseteq (\alpha + \text{Fin} (\alpha))_0 = \alpha$. 

**Corollary 11.** Nullary, elementary and finitary NID are equivalent to respectively nullary, elementary and finitary NID$^\ast$.

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